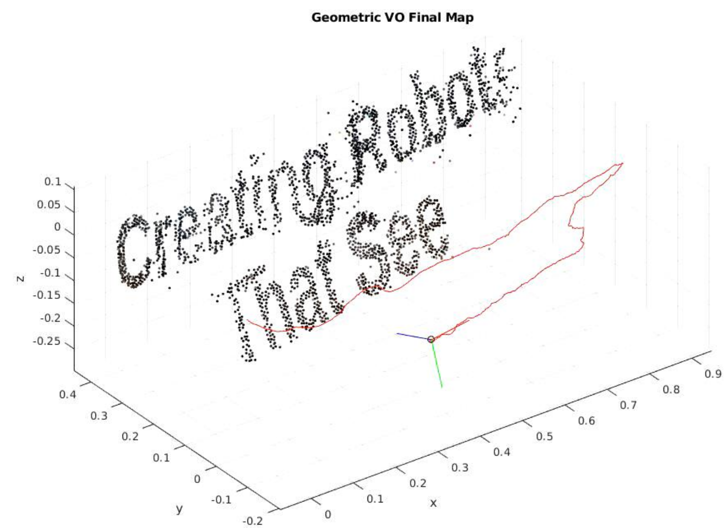
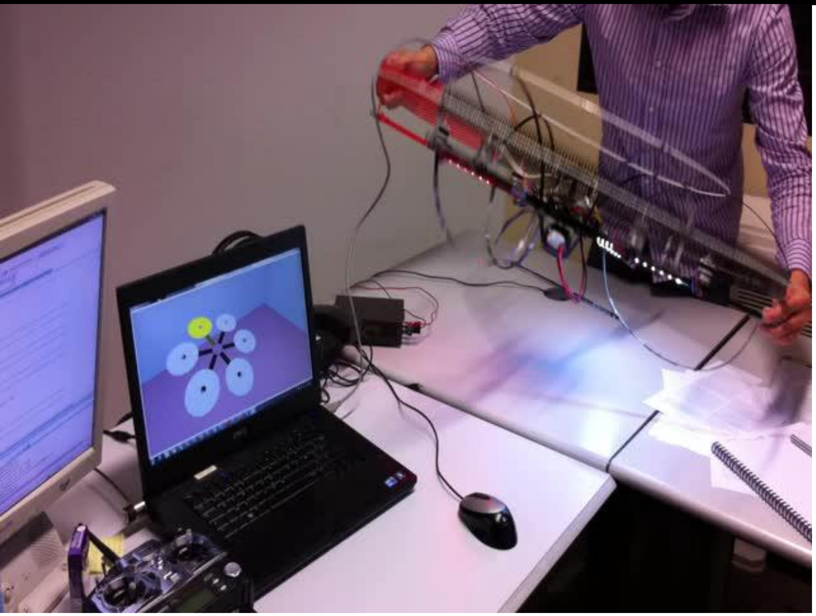
Equivariant Systems and Observer Design

Robert Mahony



IEEE Conference on Decision and Control
Semi-Plenary Wednesday 11 December

Pose estimation



Attitude and pose estimation

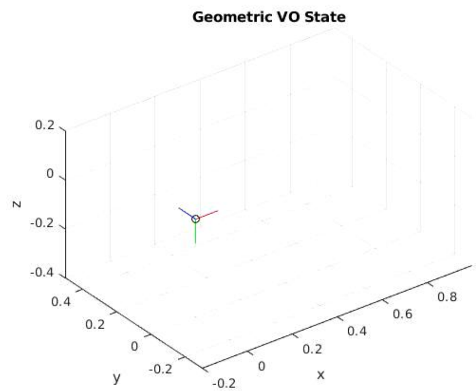
$$\hat{P} \in \mathbf{SO}(3) \text{ or } \mathbf{SE}(3)$$

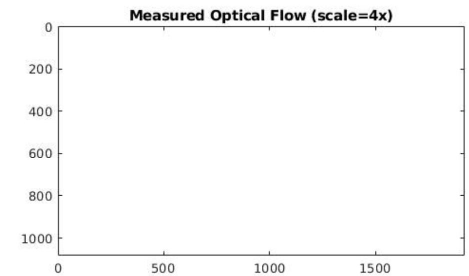
$$V \in \mathfrak{so}(3) \text{ or } \mathfrak{se}(3)$$

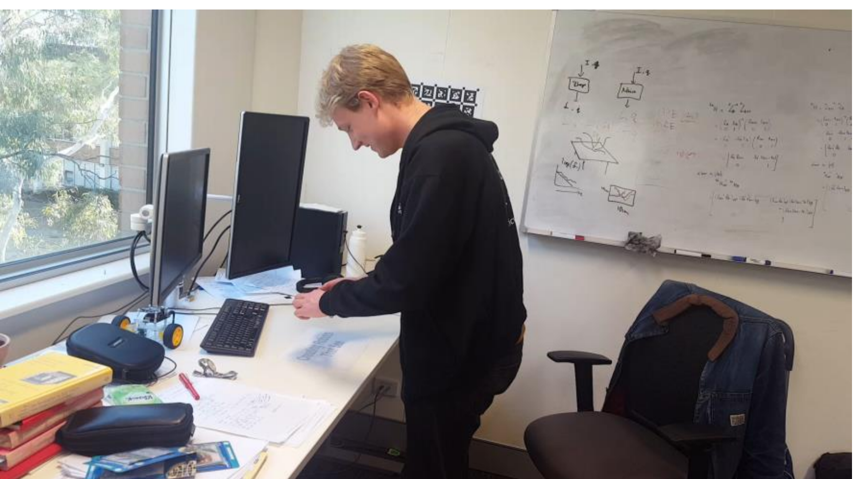
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P} = \hat{P}V - k\Delta\hat{P}$$

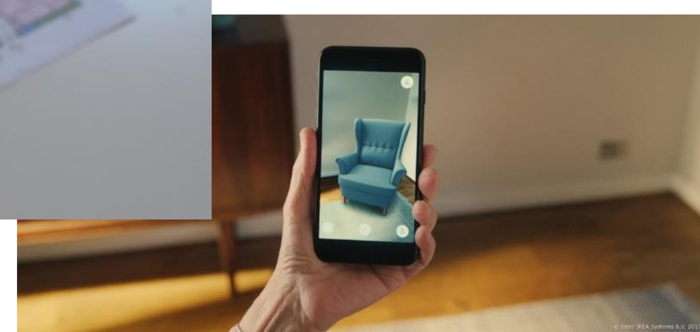
$$\Delta = \mathbb{P}_{\mathfrak{se}}\left(\sum_{i=1}^{n} k_i(\hat{P}\overline{p}_i - \overline{p}_i^{\circ})\overline{p}_i^{\top}\hat{P}^{\top}\right)$$

Spatial Awareness for Augmented Reality









Why equivariant observers

- Global analysis framework
- Global stability results
- Algebraic and algorithmic simplicity
- Low computational and memory cost
- Practical robustness to real-world measurement errors.
  - Data association errors
  - Missing data
  - False and malicious data

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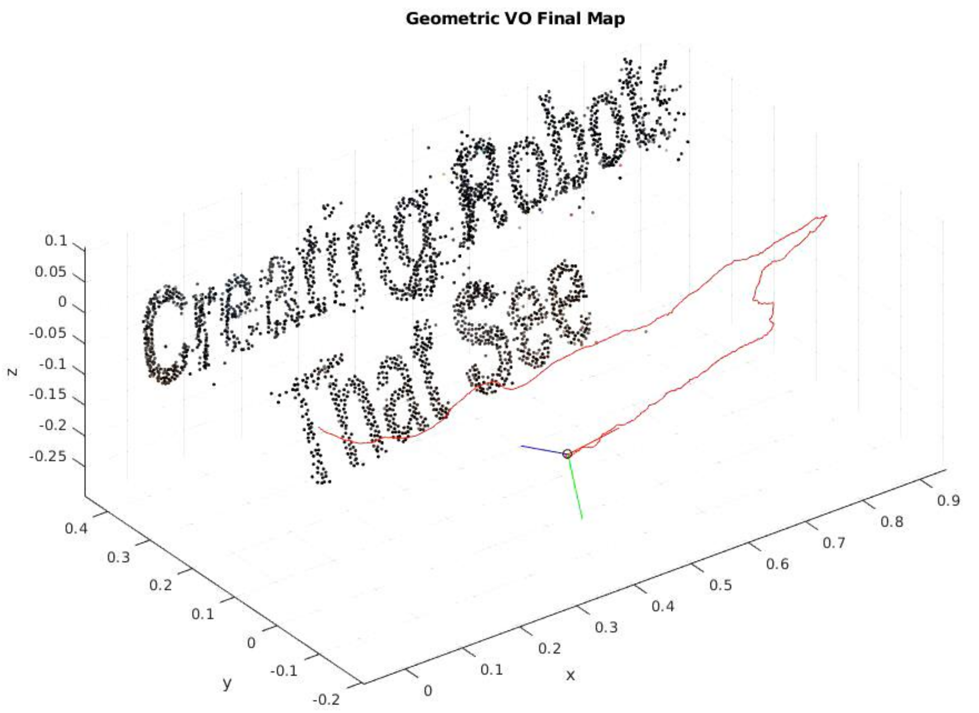
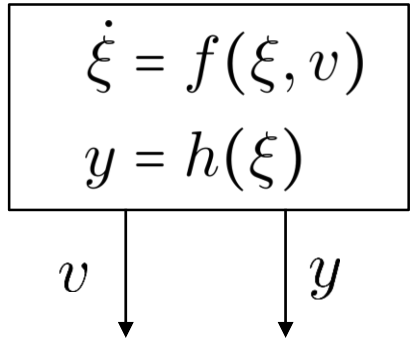
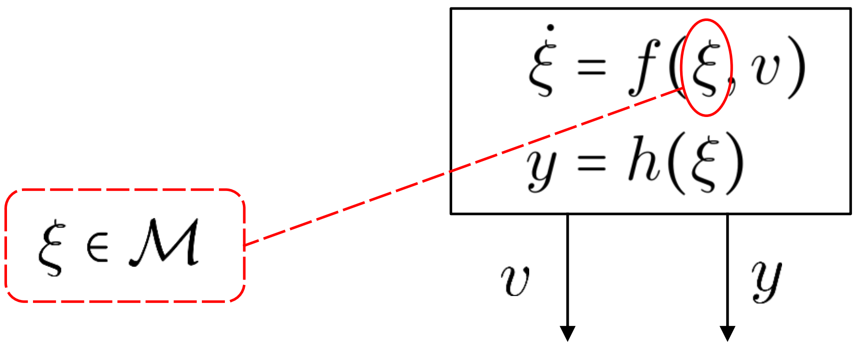


A review of nonlinear observer design

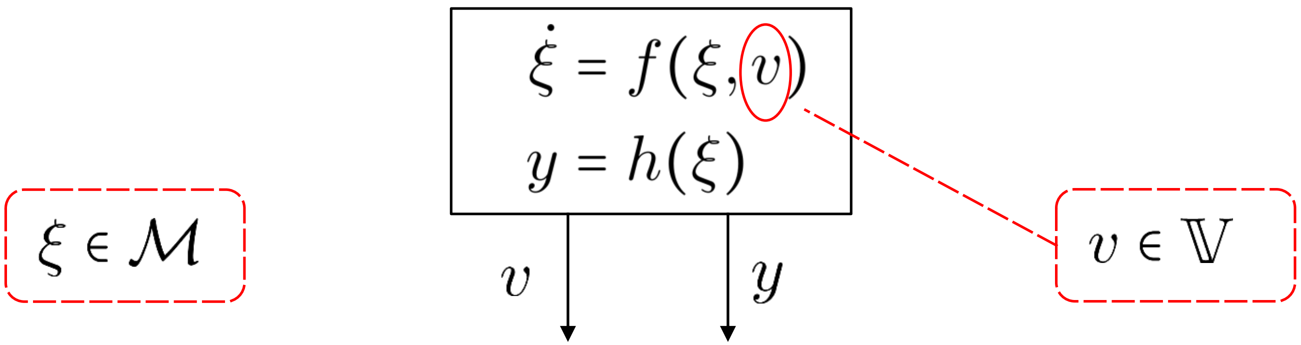
System kinematics



System kinematics



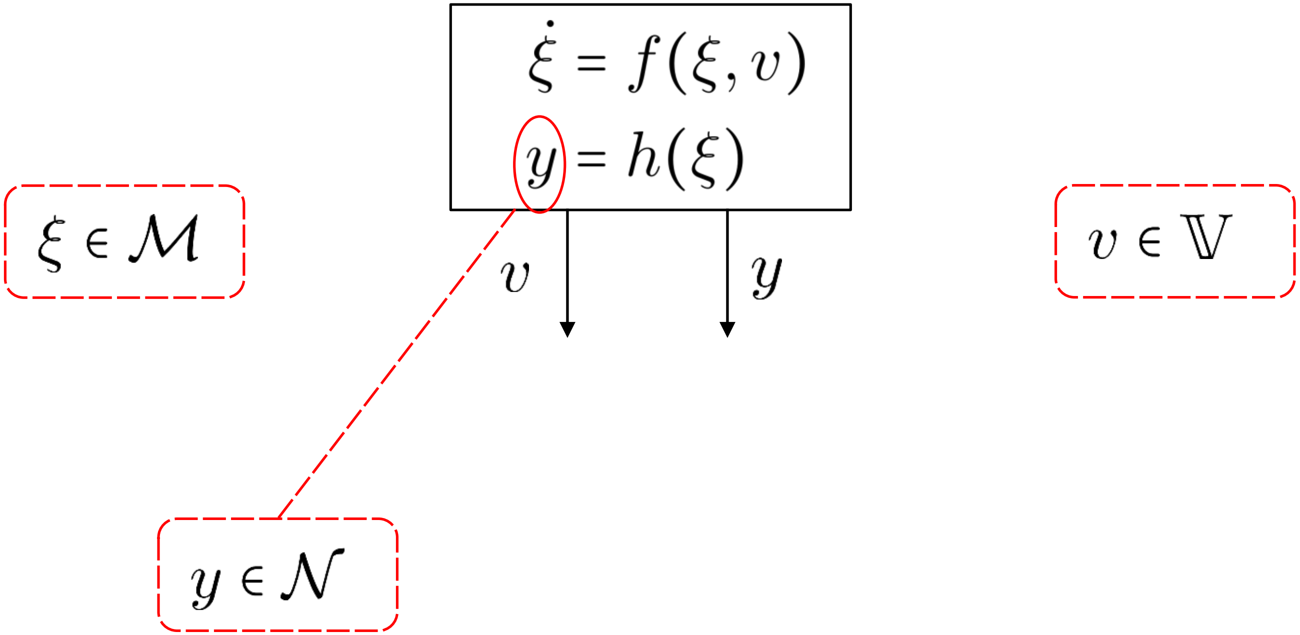
System kinematics



System function is linear in input

$$f(\xi, a_1 v_1 + a_2 v_2) = a_1 f(\xi, v_1) + a_2 f(\xi, v_2)$$

System kinematics



System kinematics

$$\dot{\xi} = f(\xi, v)$$
$$y = h(\xi)$$

v y

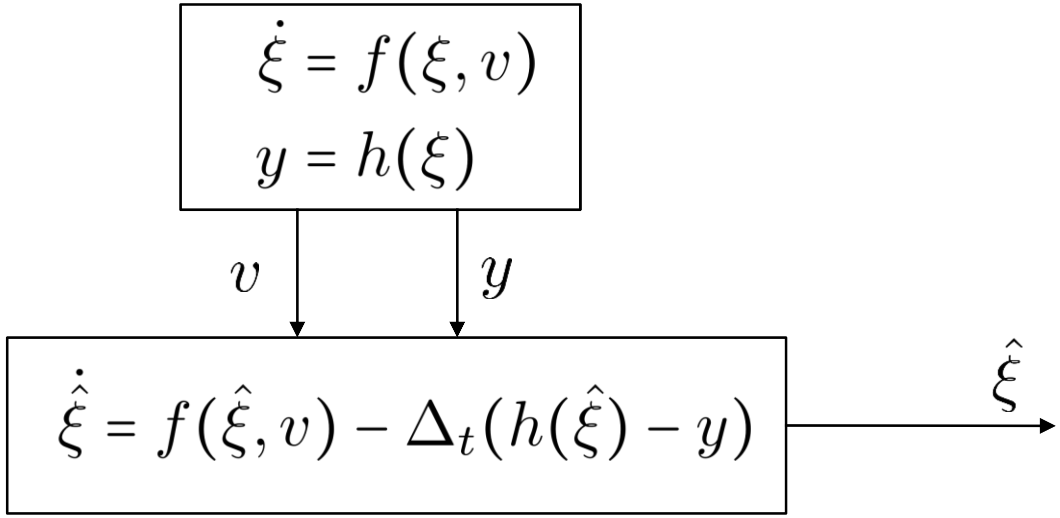
$\xi \in \mathcal{M}$

$v \in \mathbb{V}$

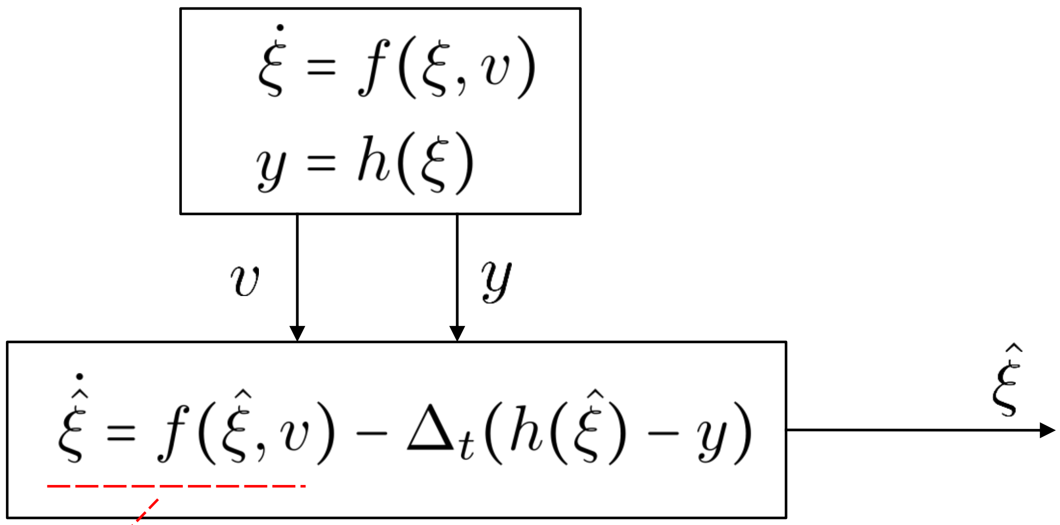
$y \in \mathcal{N}$

Measurements

Classical Observer Architecture



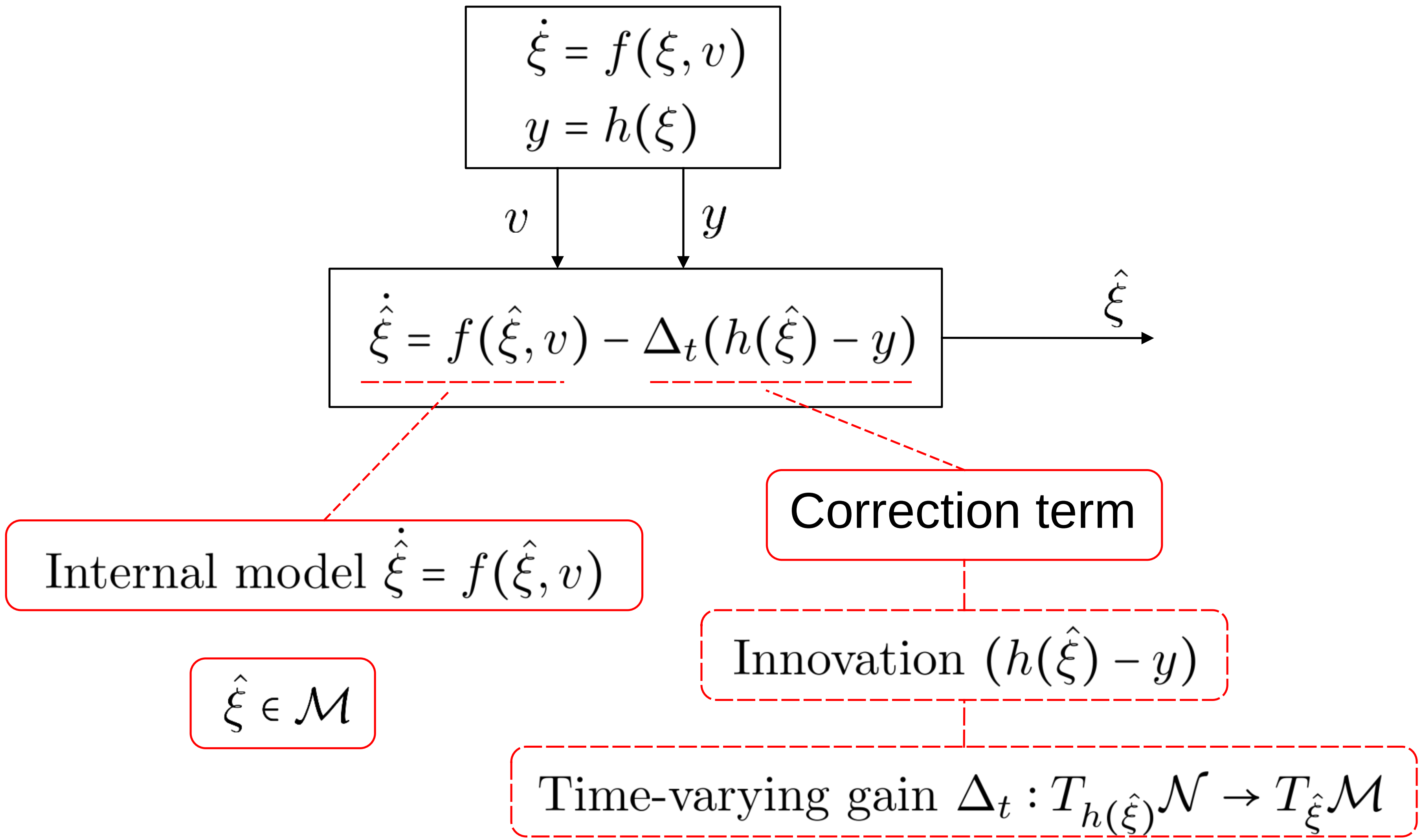
Classical Observer Architecture



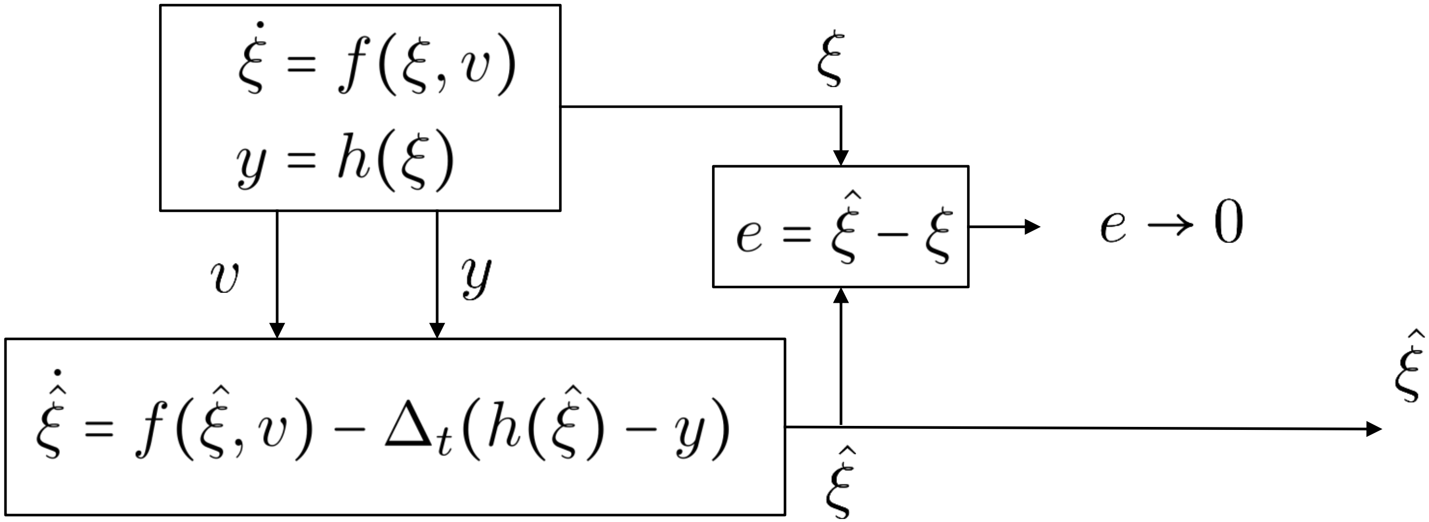
Internal model $\dot{\hat{\xi}} = f(\hat{\xi}, v)$

$\hat{\xi} \in \mathcal{M}$

Classical Observer Architecture



Observer analysis framework



A good observer design is characterised by $e(t) \to 0$

Stability

LAS - Local Asymptotic Stability

GES - Global Exponential Stability

GAS - Global Asymptotic Stability

LES – Local Exponential Stability

Practical Stability

Uniform Stability

Roadblock

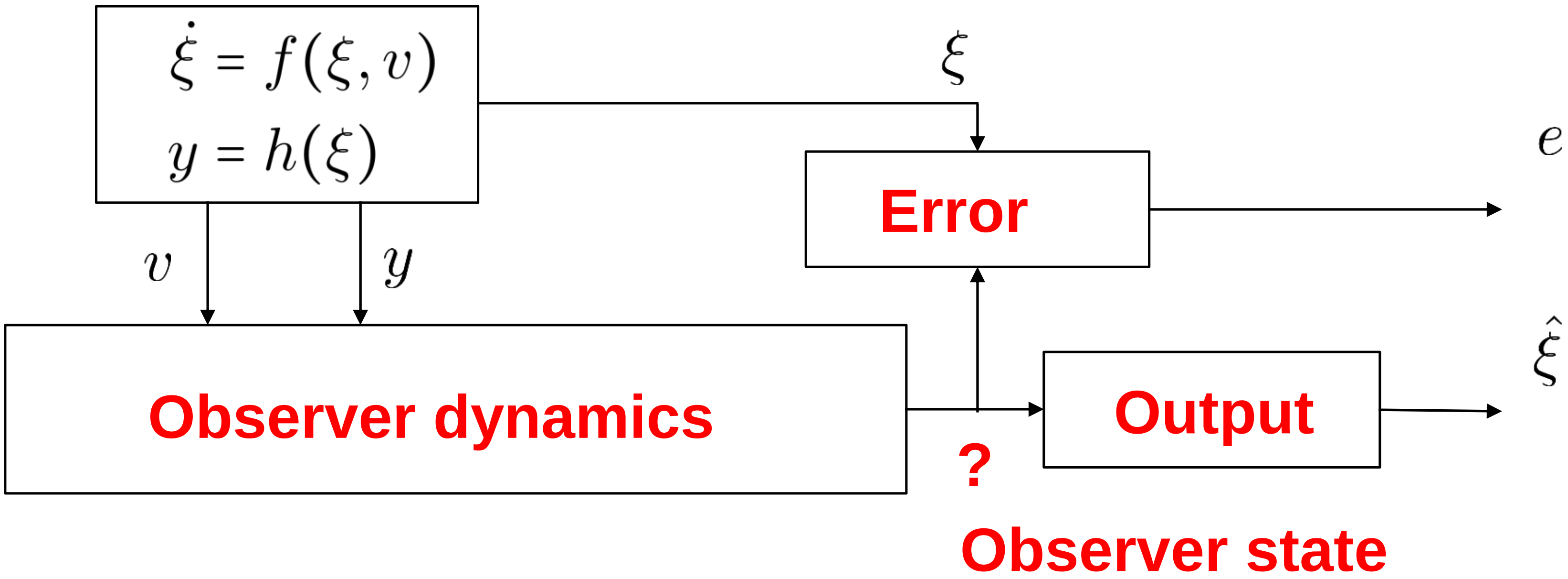
How do you compute

$$e = \hat{\xi} - \xi$$

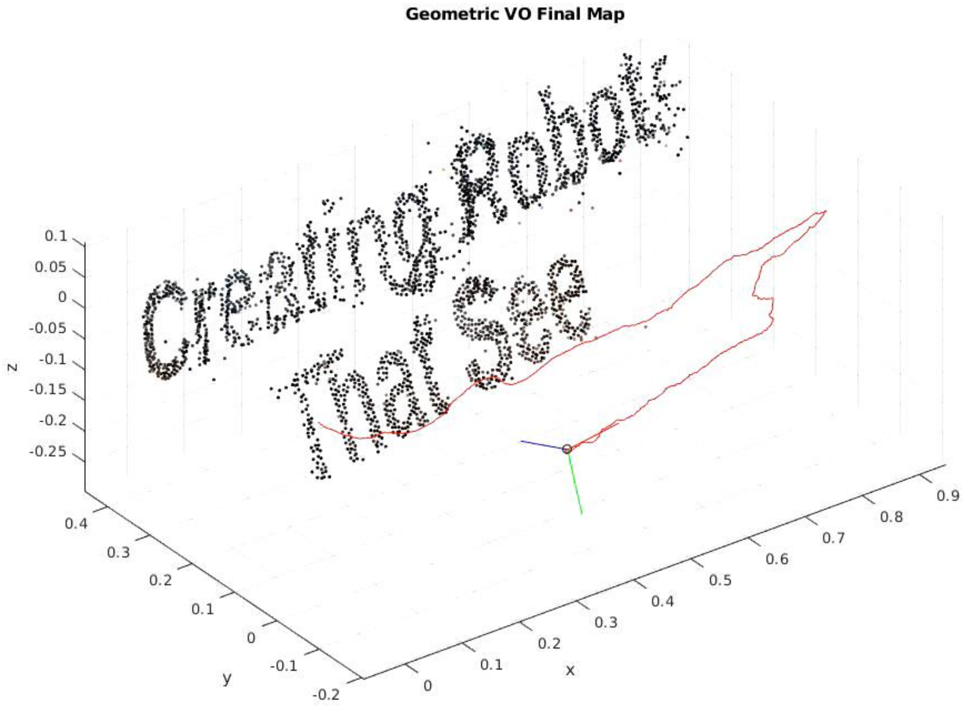
on a manifold!

- How do you compute the innovation $h(\hat{\xi}) - y$?
- Why is the state space of the observer $\hat{\xi} \in \mathcal{M}$?

Equivariant observer architecture



- A new observer state space
- An output map from the new state to the desired estimate
- A well defined global error signal
- Observer dynamics (internal model and correction term)

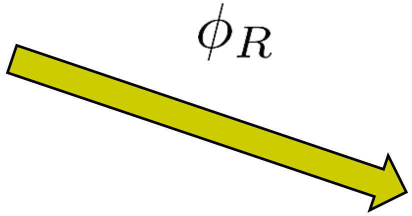


An introduction to symmetry

Symmetry

An apple is a manifold

A symmetry is a mapping that preserves the structure of the space



A rotation matrix $R \in \mathbf{SO}(3)$

$$\phi_R : \mathrm{S}^2 \to \mathrm{S}^2$$

$$\phi_R(\xi) := R^\top \xi$$

Parametrizing state by symmetry

Define

$$\phi_{\xi^\circ} : \mathbf{SO}(3) \to \mathrm{S}^2$$

$$\phi_{\xi^\circ}(R) := R^\top \xi^\circ = \phi_R(\xi^\circ)$$

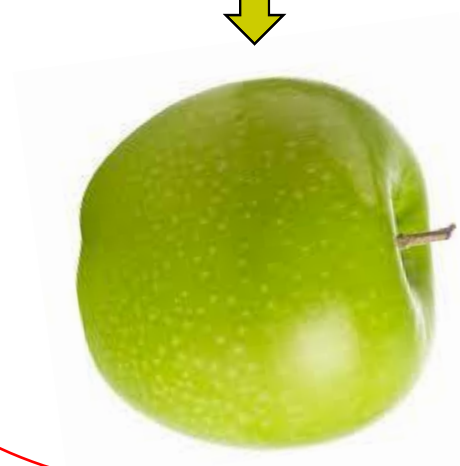
$\mathbf{SO}(3)$

ϕ_{ξ°

Fix an *origin* ξ°

ξ°

ϕ_R

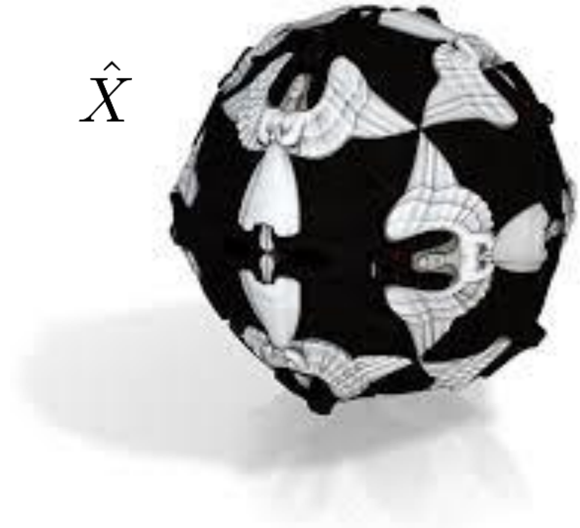


$\xi = R^\top \xi^\circ$

Group parametrization

Observer state $\hat{X} \in \mathbf{G}$

$\hat{X}$

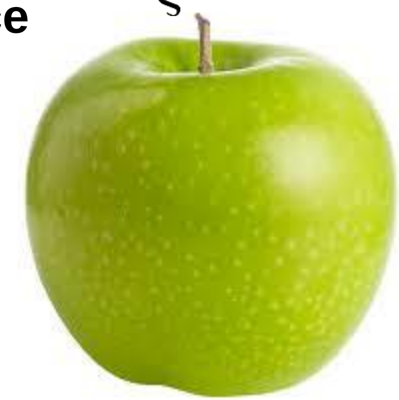


$\mathbf{G}$

Lie group

State space

ξ°



ξ

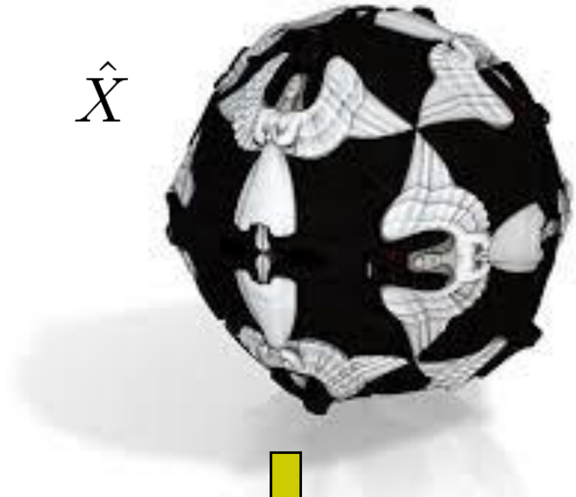
$\mathcal{M}$

Group parametrization

Observer state $\hat{X} \in \mathbf{G}$

Observer output $\hat{\xi} \in \mathcal{M}$

$\hat{X}$



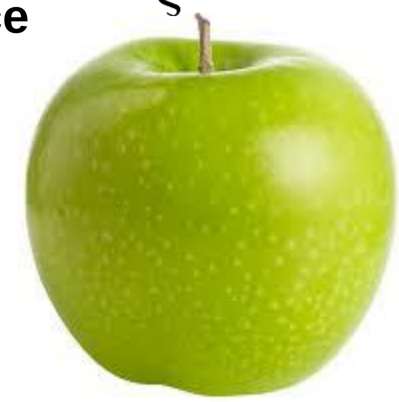
$\mathbf{G}$

Lie group

$$\hat{\xi} = \phi_{\xi^\circ}(\hat{X})$$

State space

ξ°



ξ

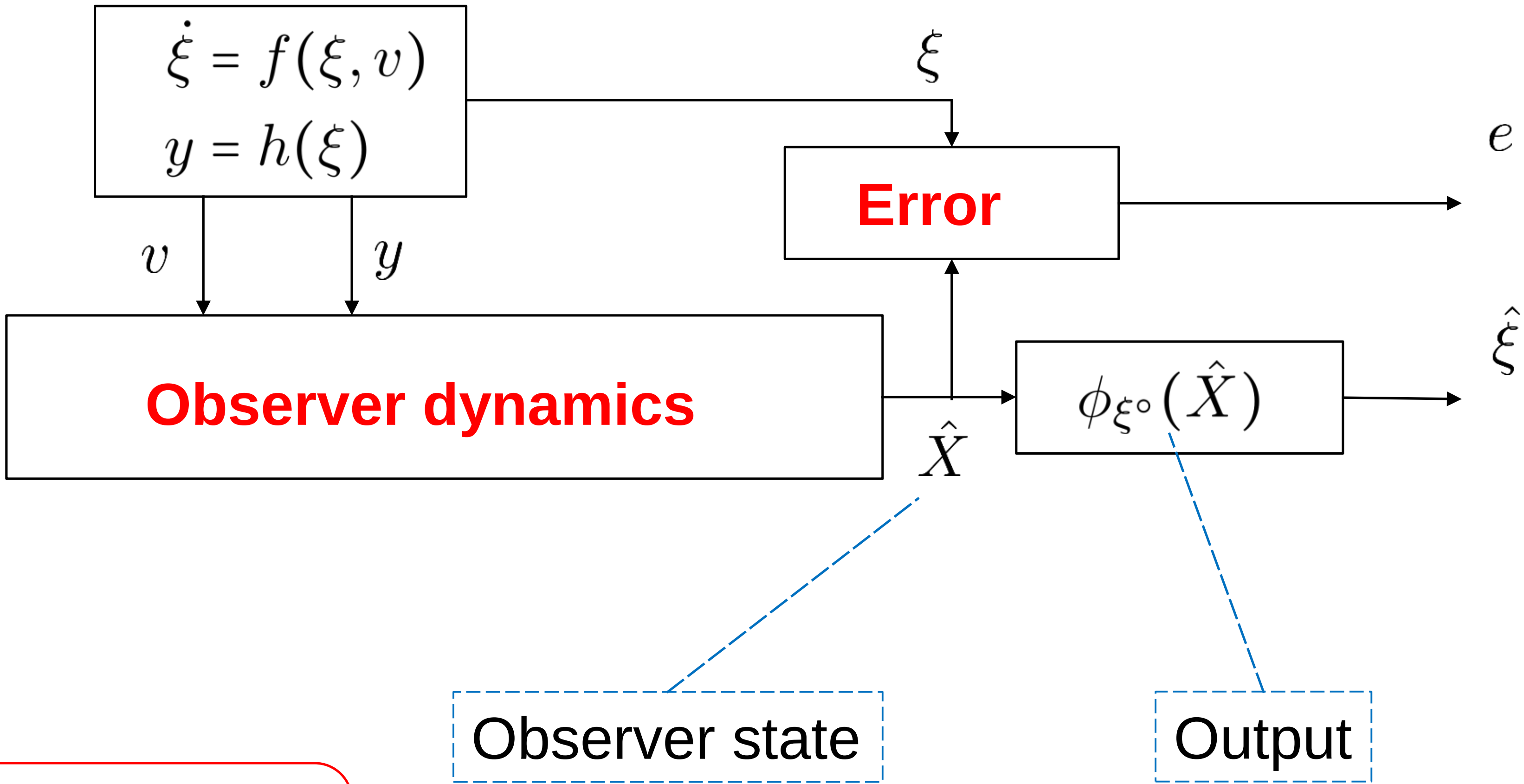
$\mathcal{M}$

Observer output

$\hat{\xi}$

$\mathcal{M}$

Observer state space and output



$\hat{X} \in \mathbf{G}$

$\hat{\xi} = \phi_{\xi^\circ}(\hat{X}) \in \mathcal{M}$

Fix an origin

$\xi^\circ \in \mathcal{M}$

Error

If $e = \xi^\circ$ then

$$\hat{\xi} = \phi_{\hat{X}}(\xi^\circ) = \phi_{\hat{X}}(e) = \xi$$

$\hat{X}$

G

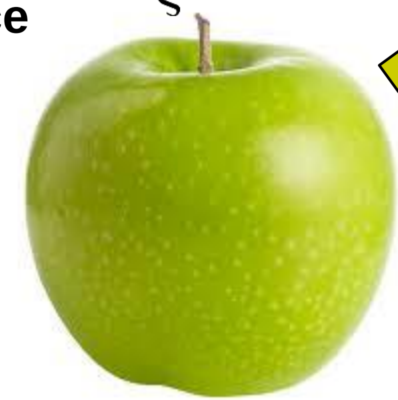
Lie group

$$e := \phi_{\hat{X}^{-1}}(\xi)$$

$$\hat{\xi} = \phi_{\xi^\circ}(\hat{X})$$

State space

ξ°



ξ

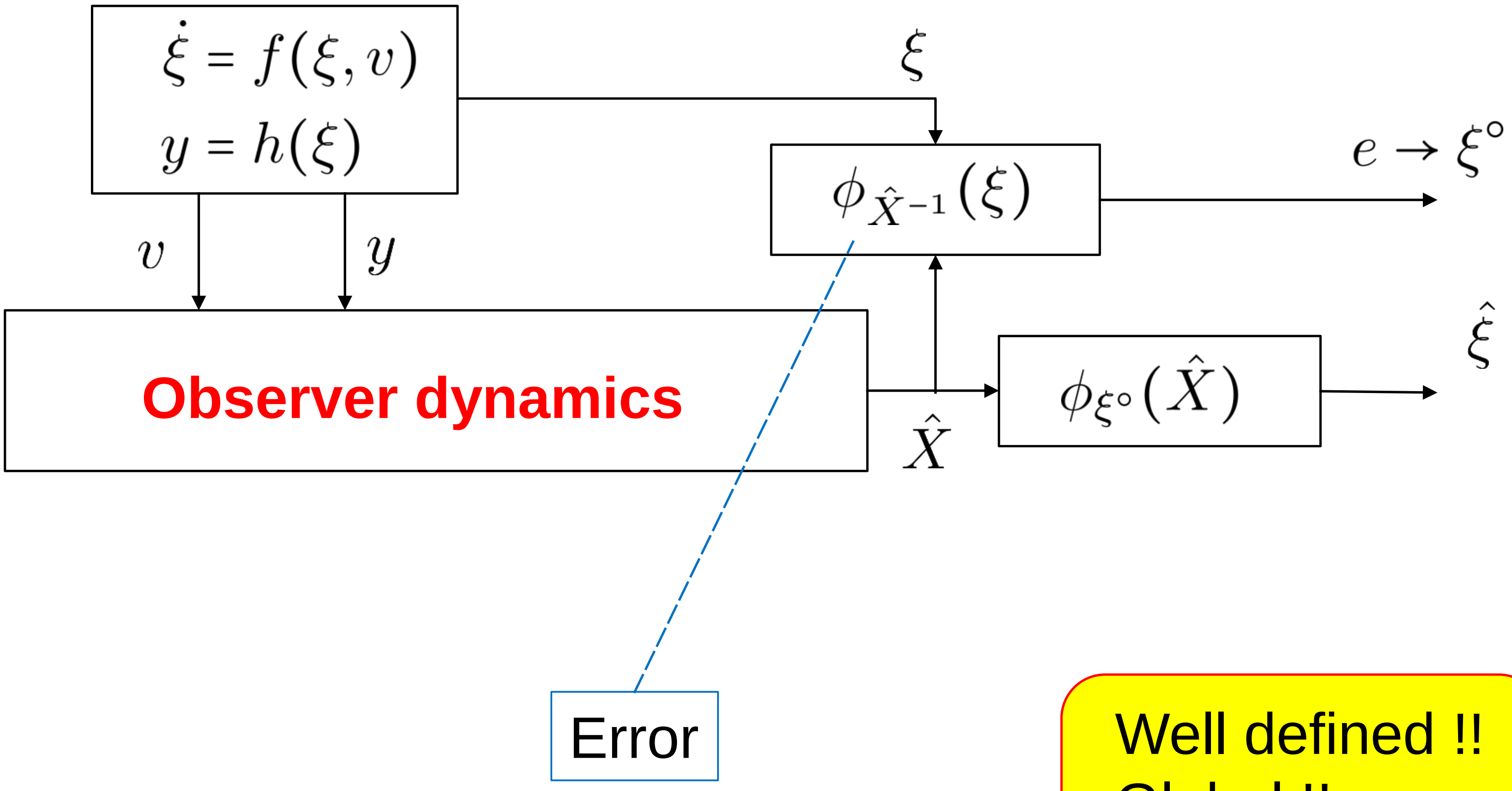
$\mathcal{M}$

$\hat{\xi}$

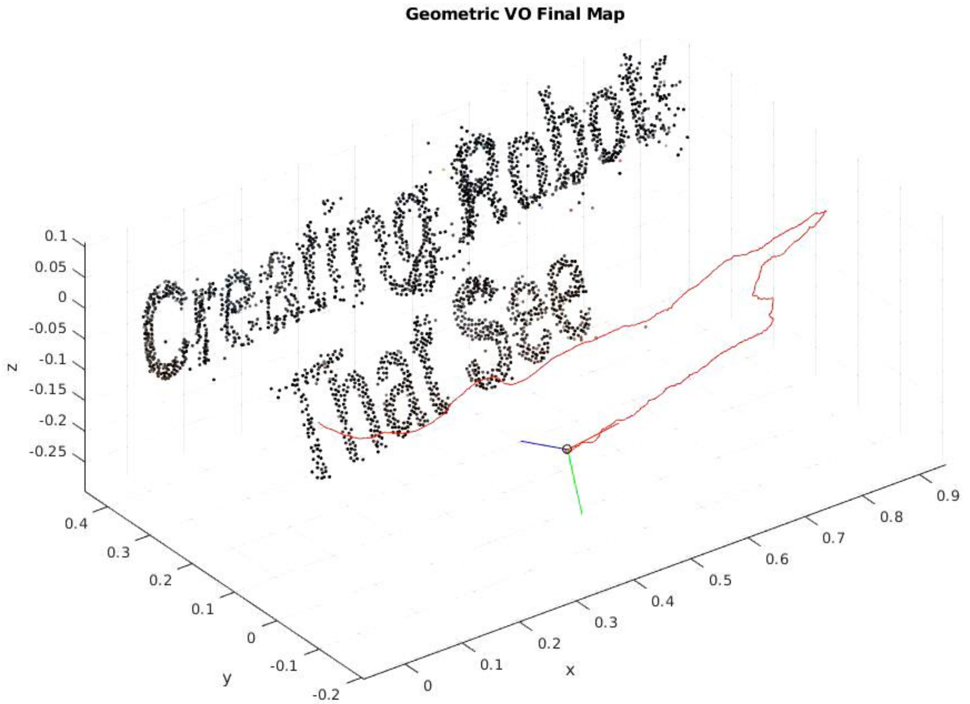
Observer output

$\mathcal{M}$

Observer state space and output



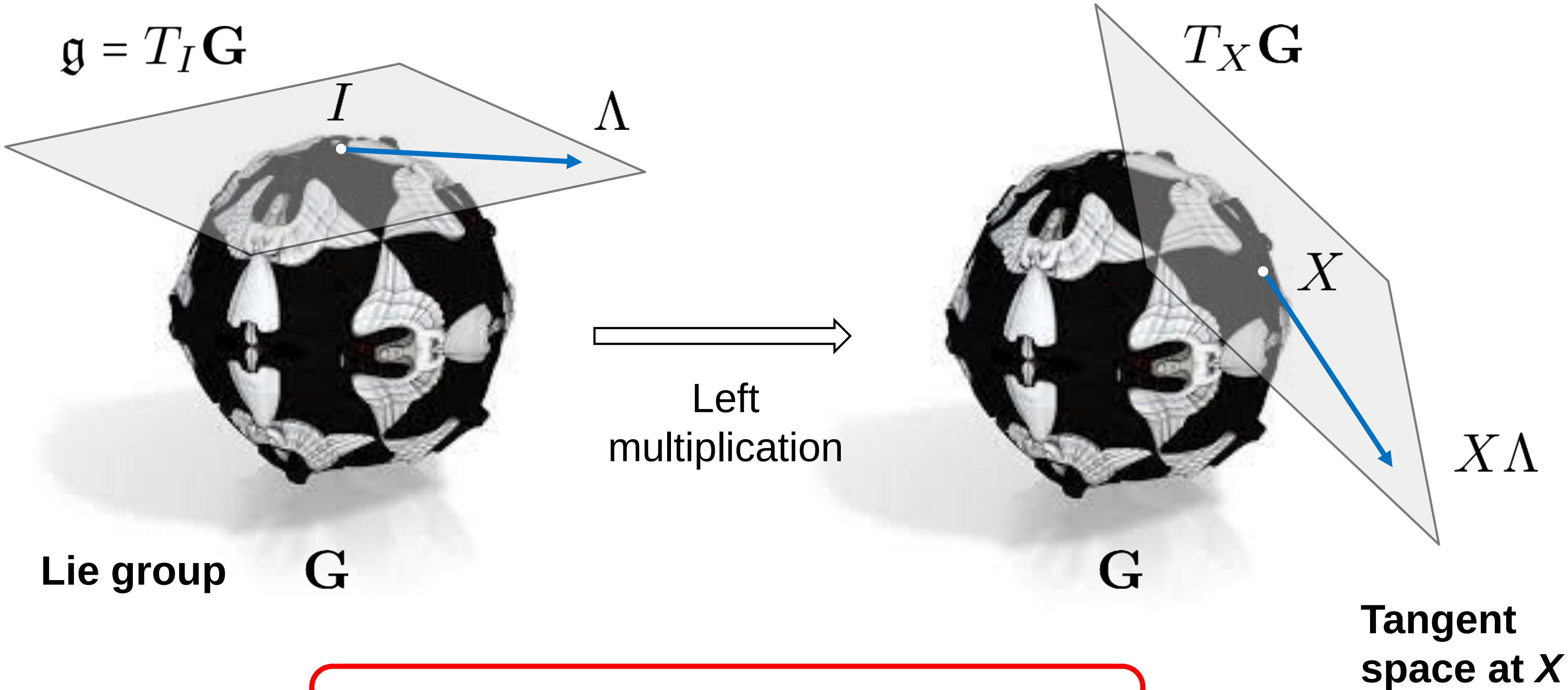
Well defined !!
Global !!
Smooth !!



Lifted System and Internal Model

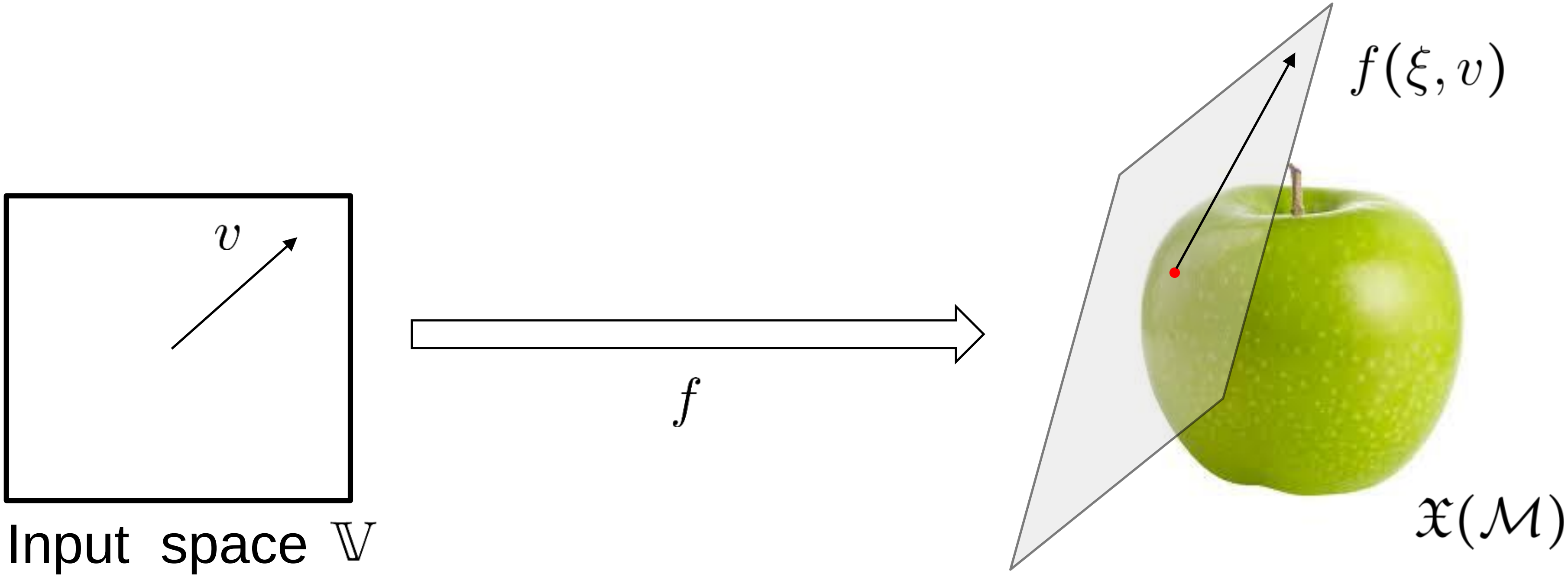
Tangent spaces

Lie algebra = tangent space at identity

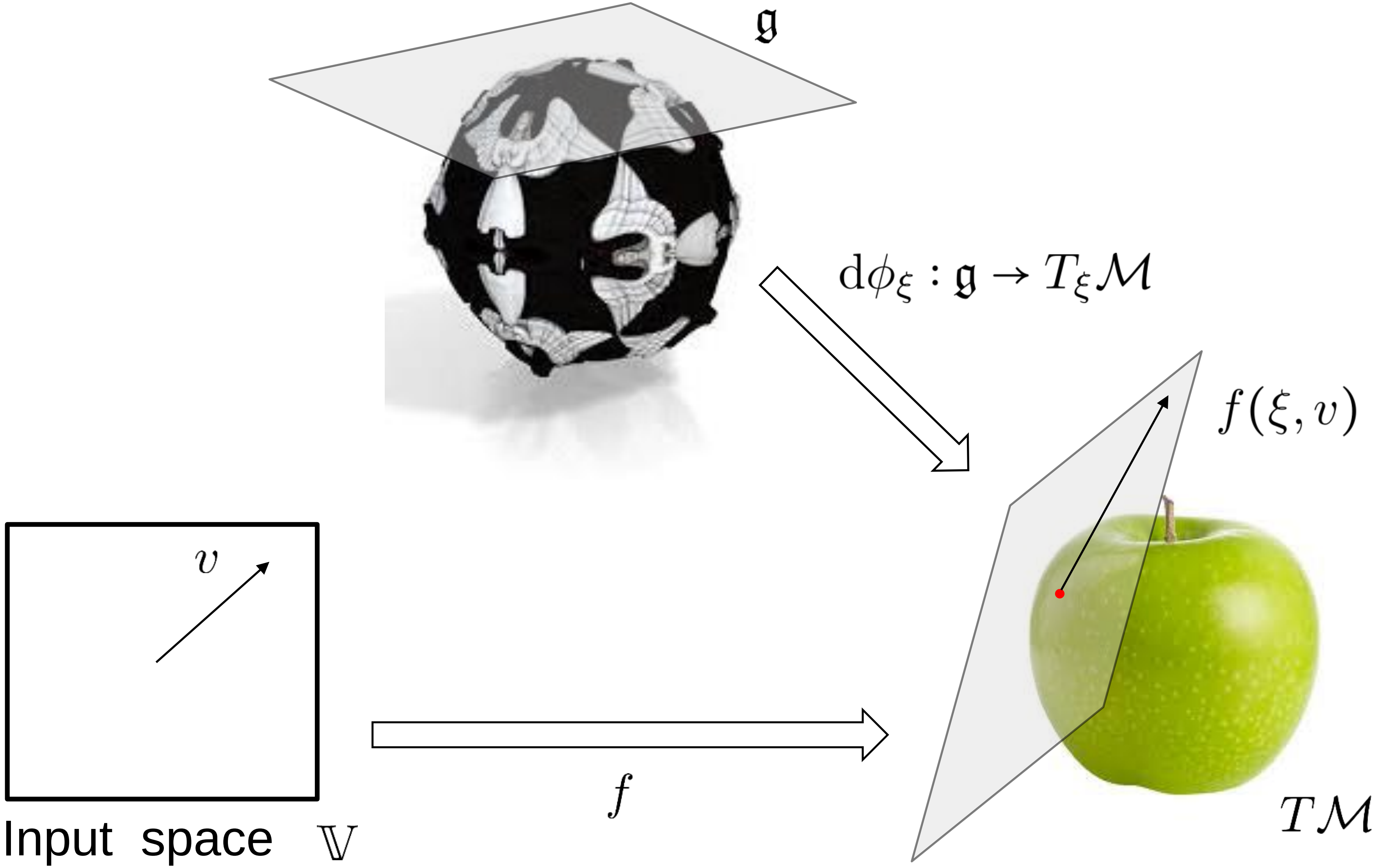


Kinematics on $\mathbf{G}$ are $\dot{X} = X\Lambda(t)$

Lift: System kinematics

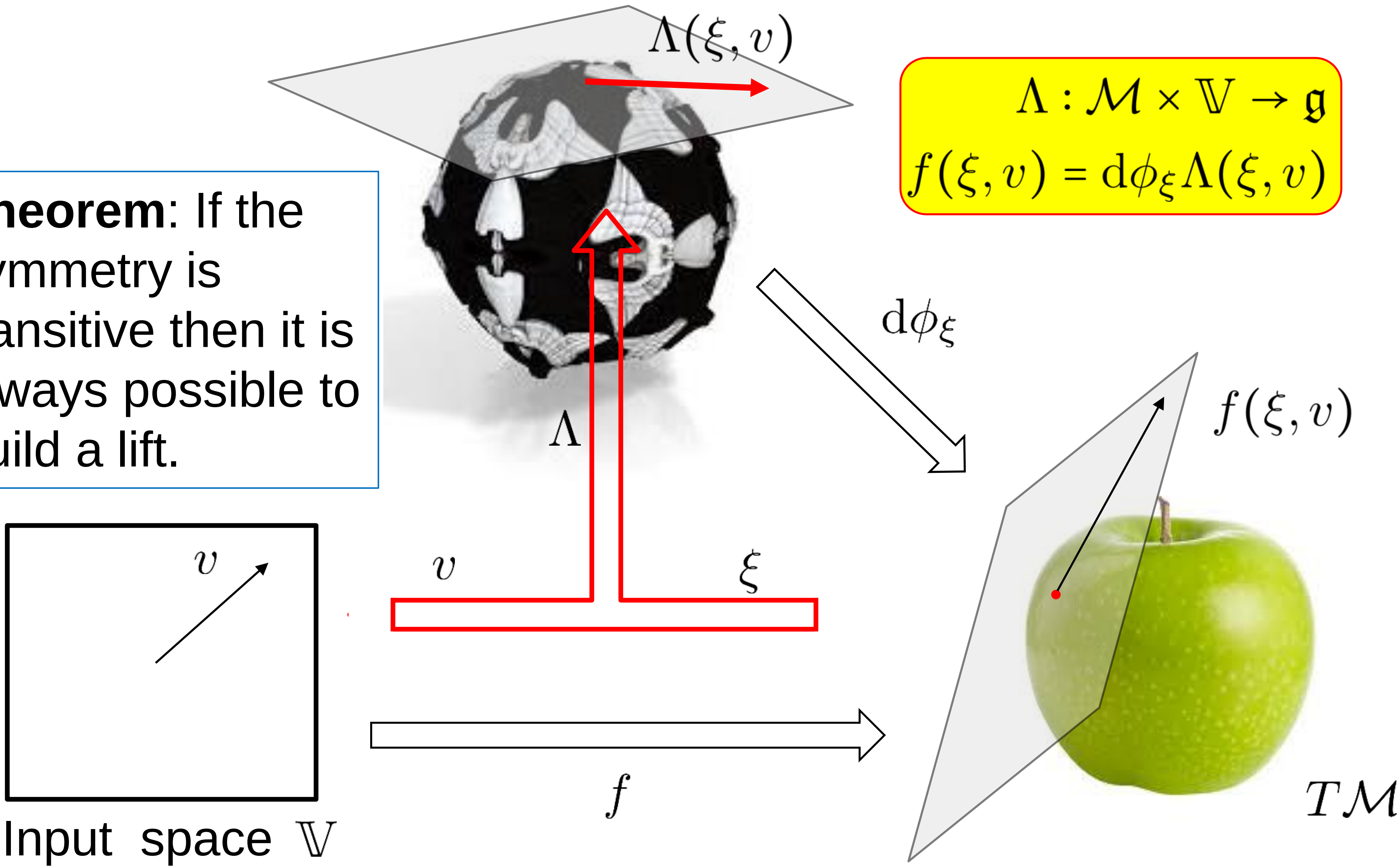


Lift: Action projection

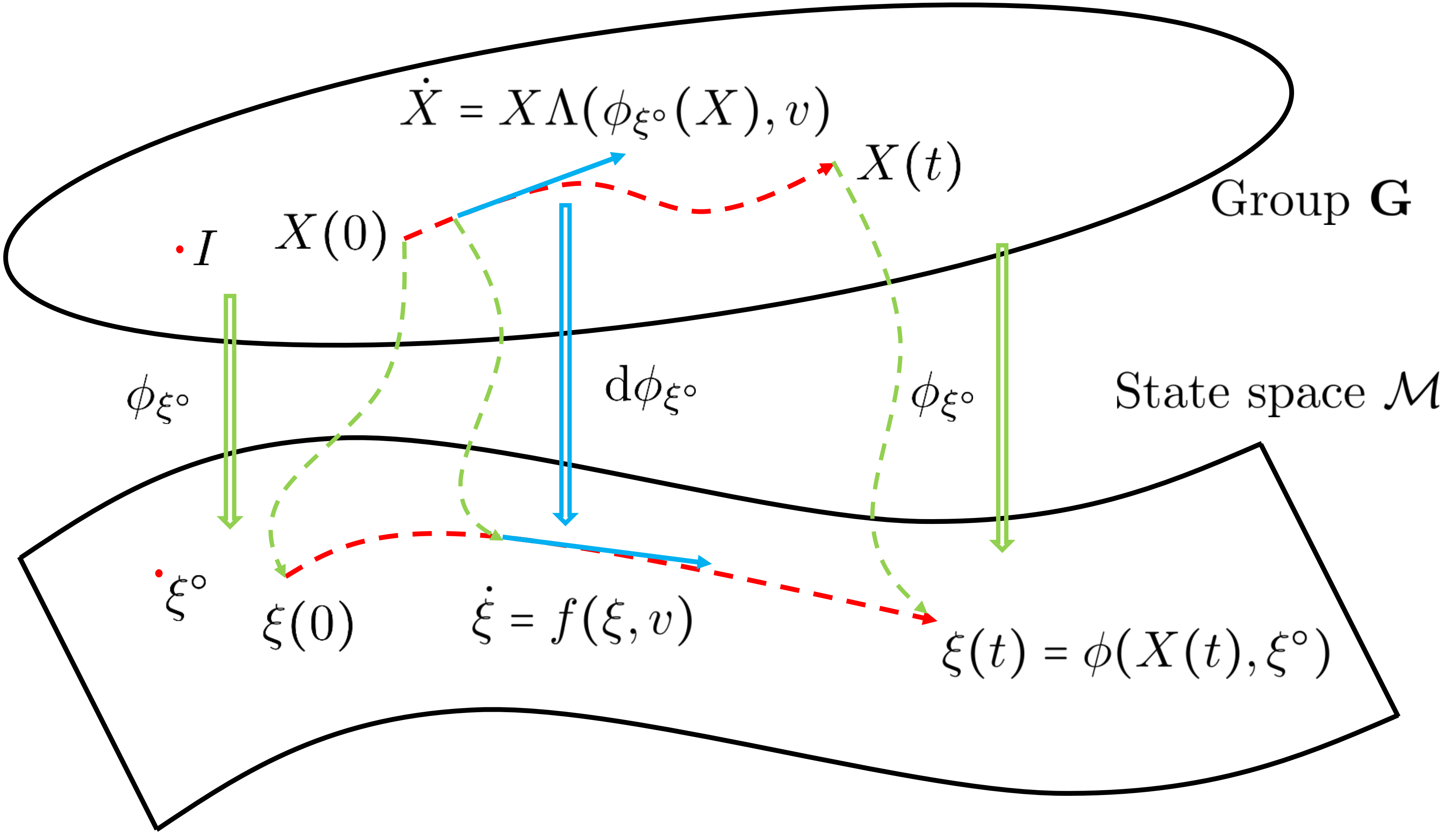


Lift: Finding a right inverse

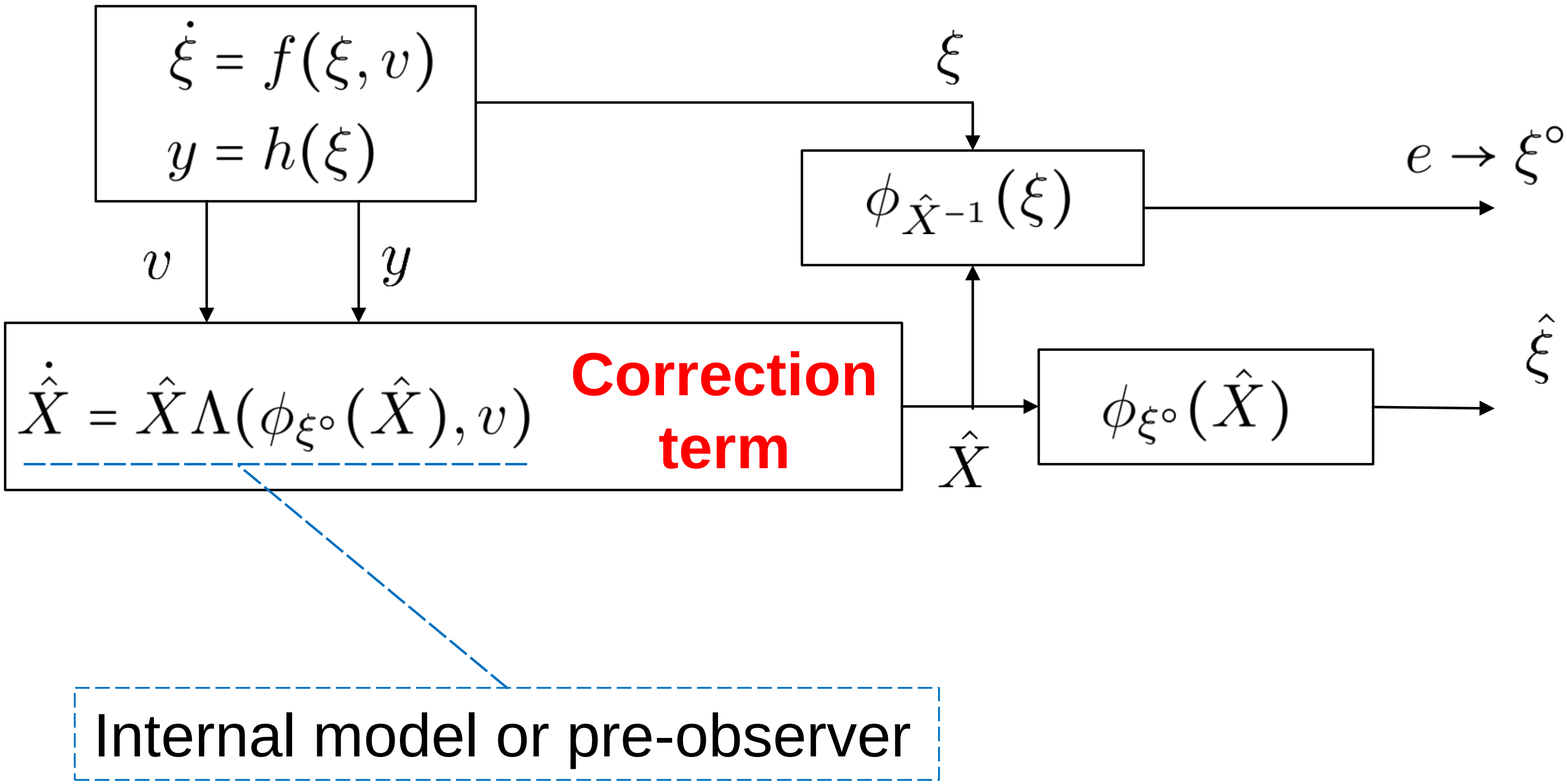
Theorem: If the symmetry is transitive then it is always possible to build a lift.

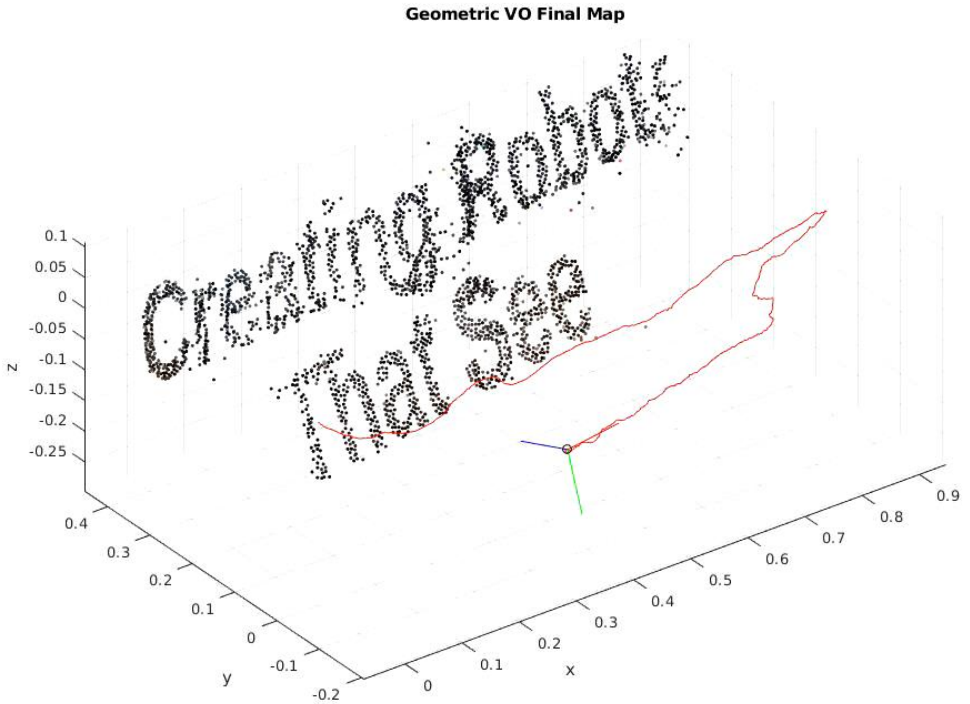


Theorem


$\dot{X} = X\Lambda(\phi_{\xi^\circ}(X), v)$
$X(t)$
Group $\mathbf{G}$
I
$X(0)$
ϕ_{ξ°
$\mathrm{d}\phi_{\xi^\circ}$
ϕ_{ξ°
State space $\mathcal{M}$
ξ°
$\xi(0)$
$\dot{\xi} = f(\xi, v)$
$\xi(t) = \phi(X(t), \xi^\circ)$

Internal model





Output Symmetry and Innovations

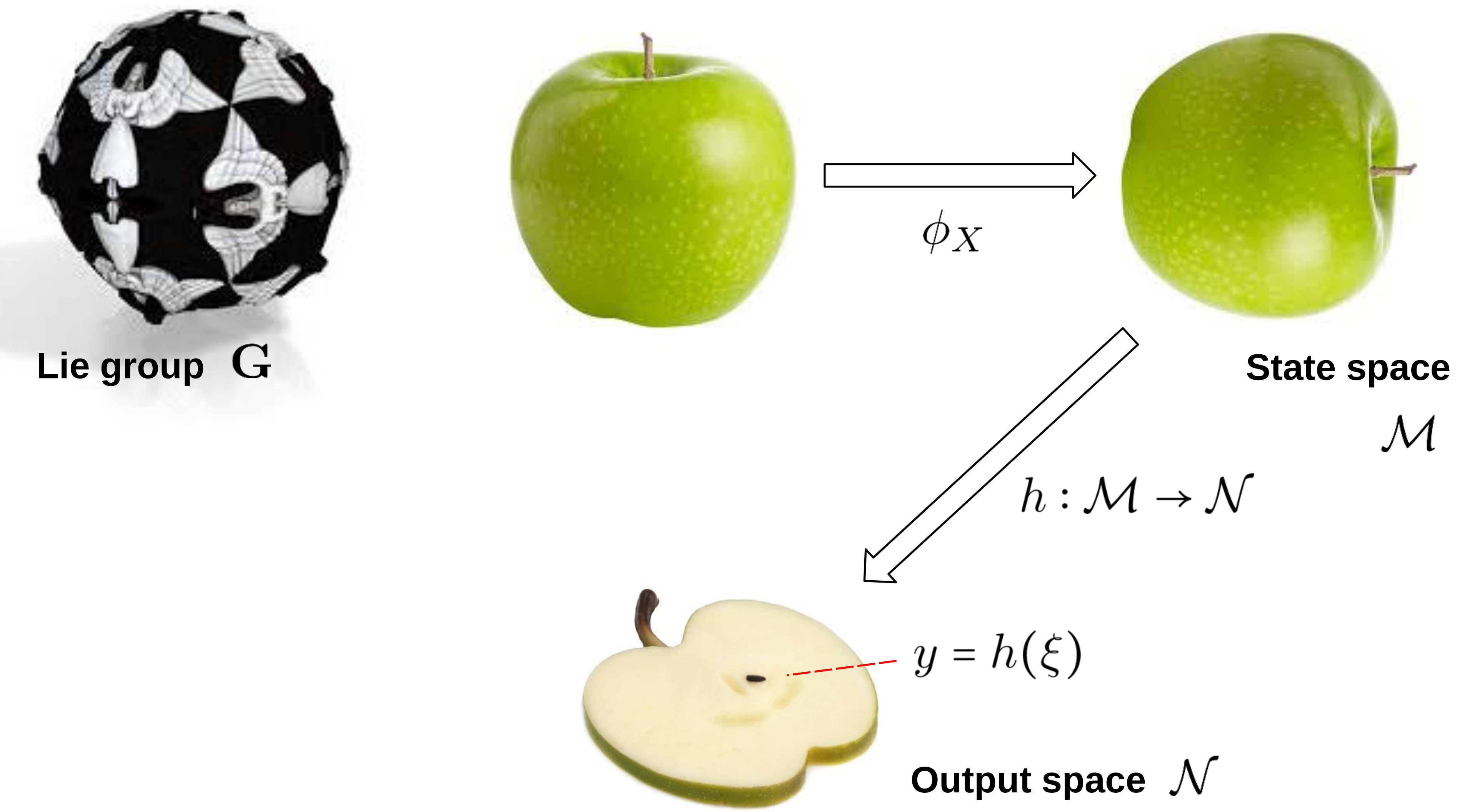
Start with state symmetry

Lie group $\mathbf{G}$

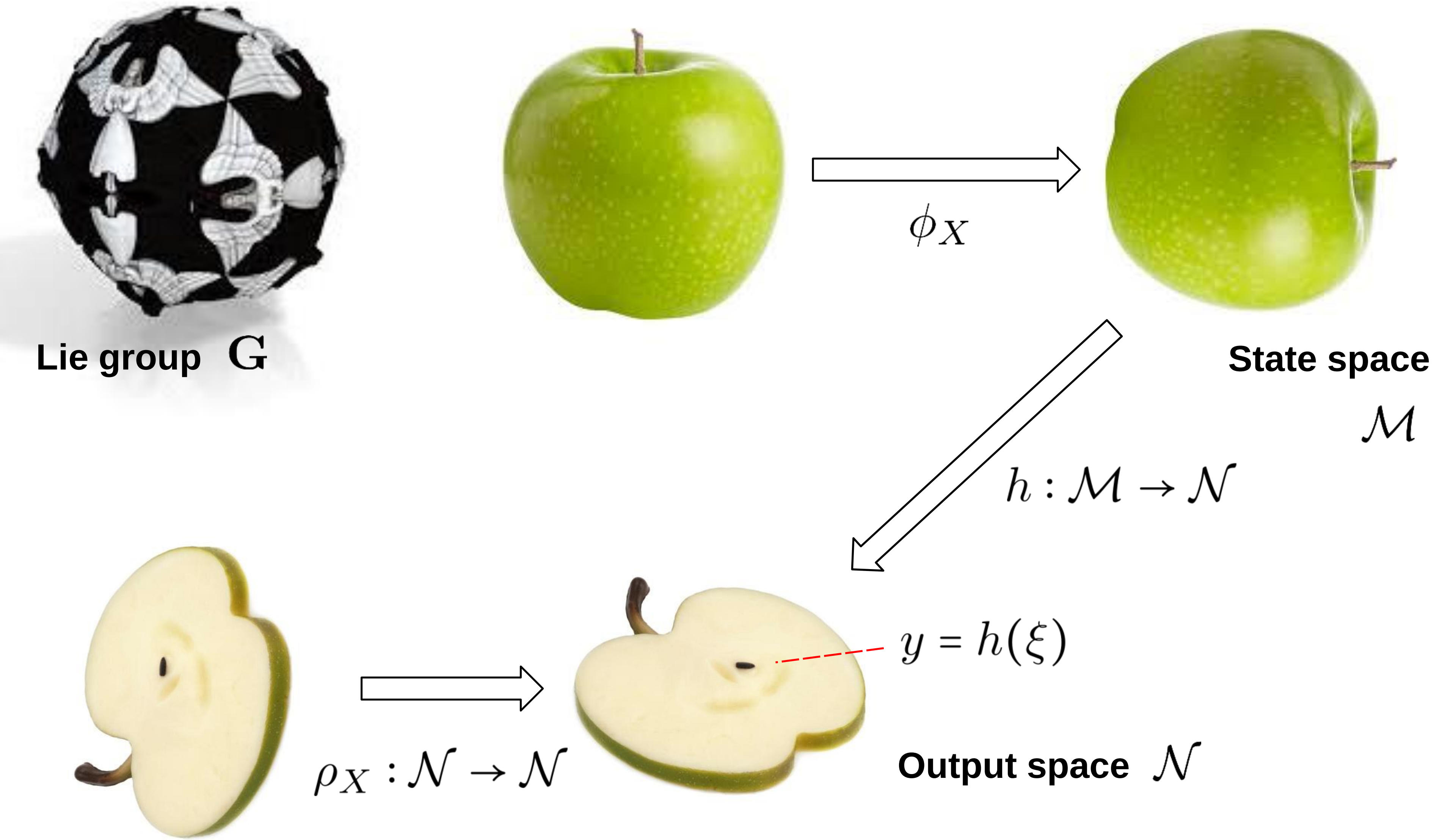
ϕ_X

State space $\mathcal{M}$

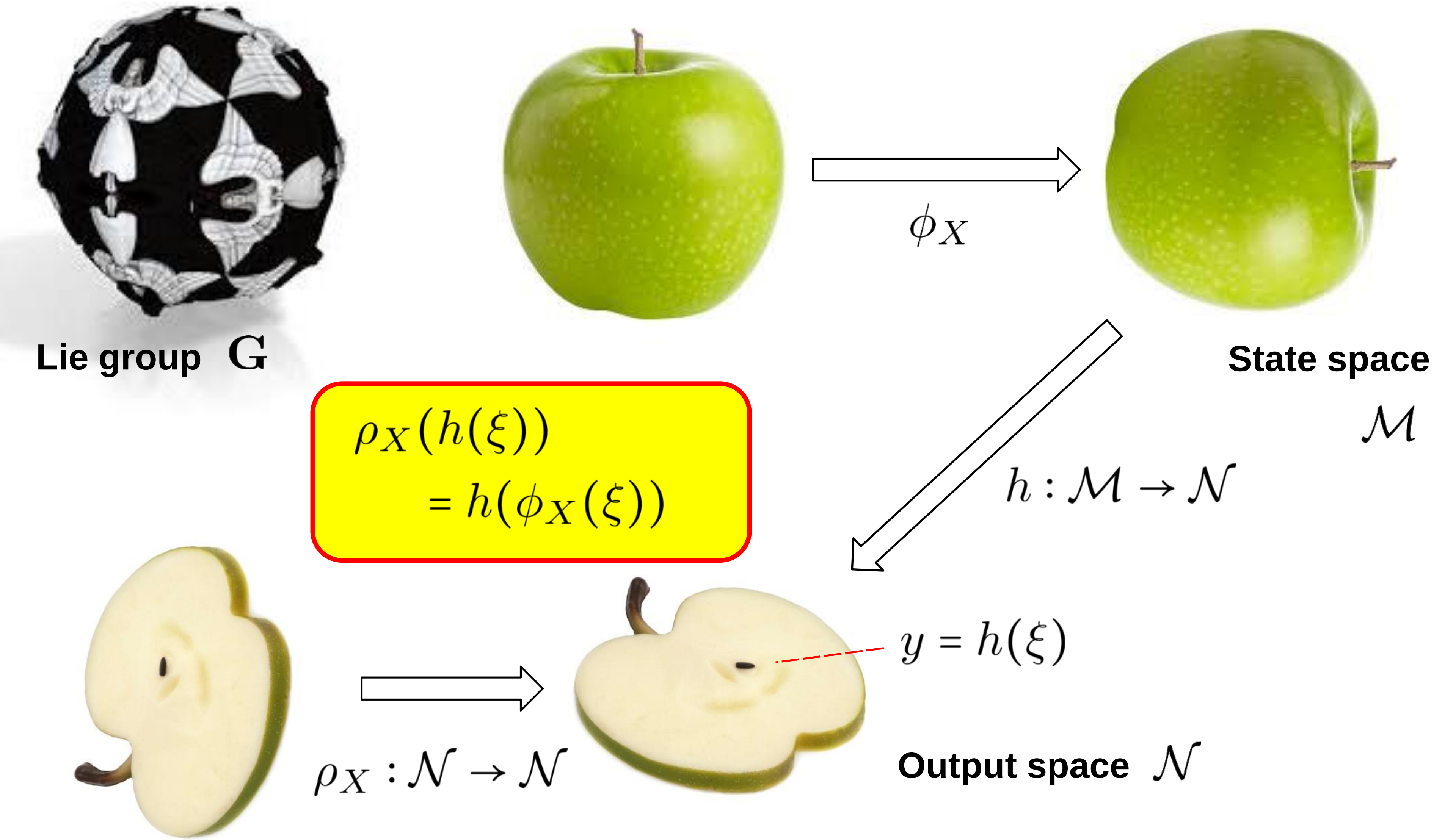
Output map


Lie group $\mathbf{G}$
ϕ_X
State space
$\mathcal{M}$
$h : \mathcal{M} \to \mathcal{N}$
$y = h(\xi)$
Output space $\mathcal{N}$

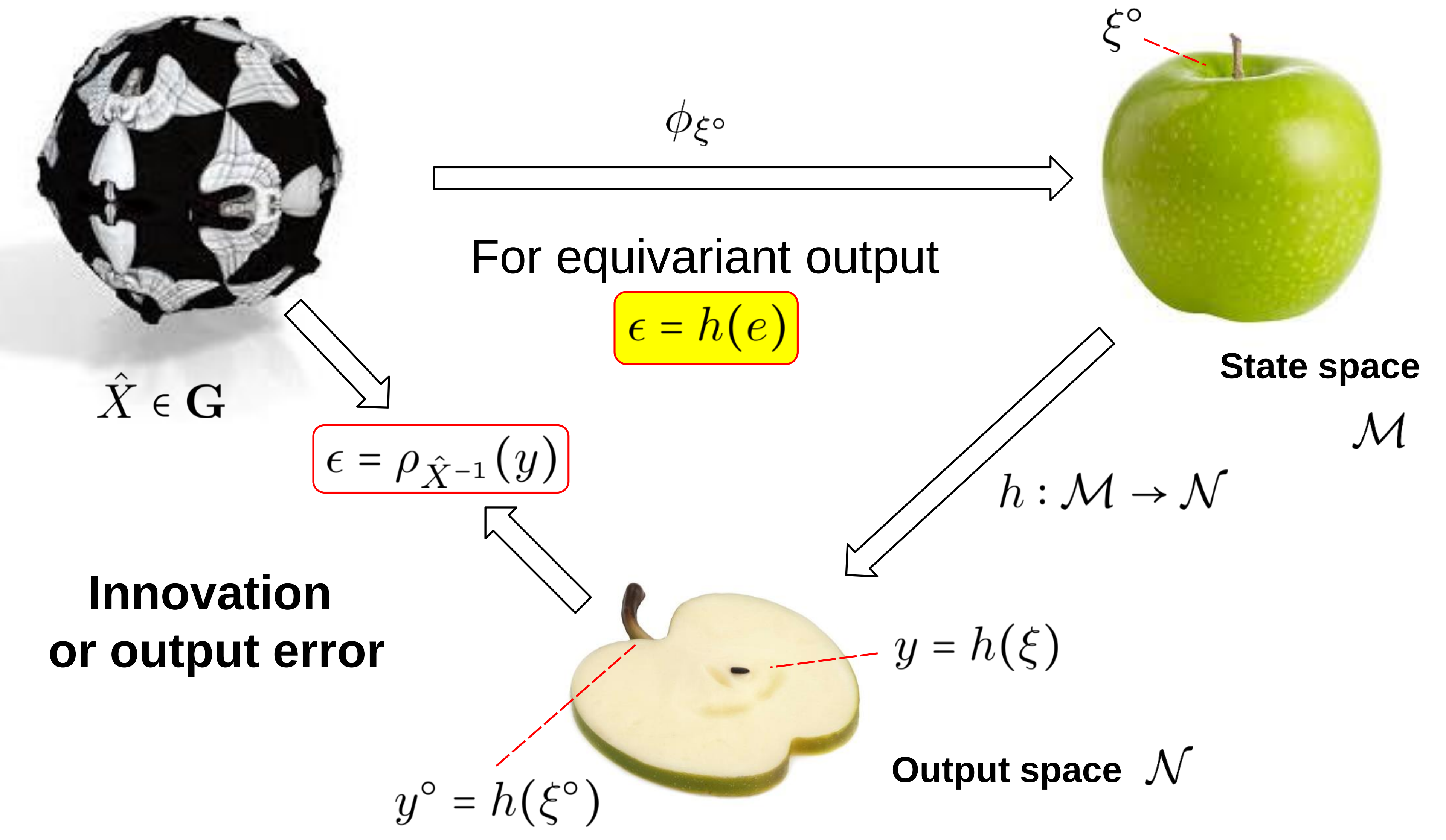
Output Symmetry



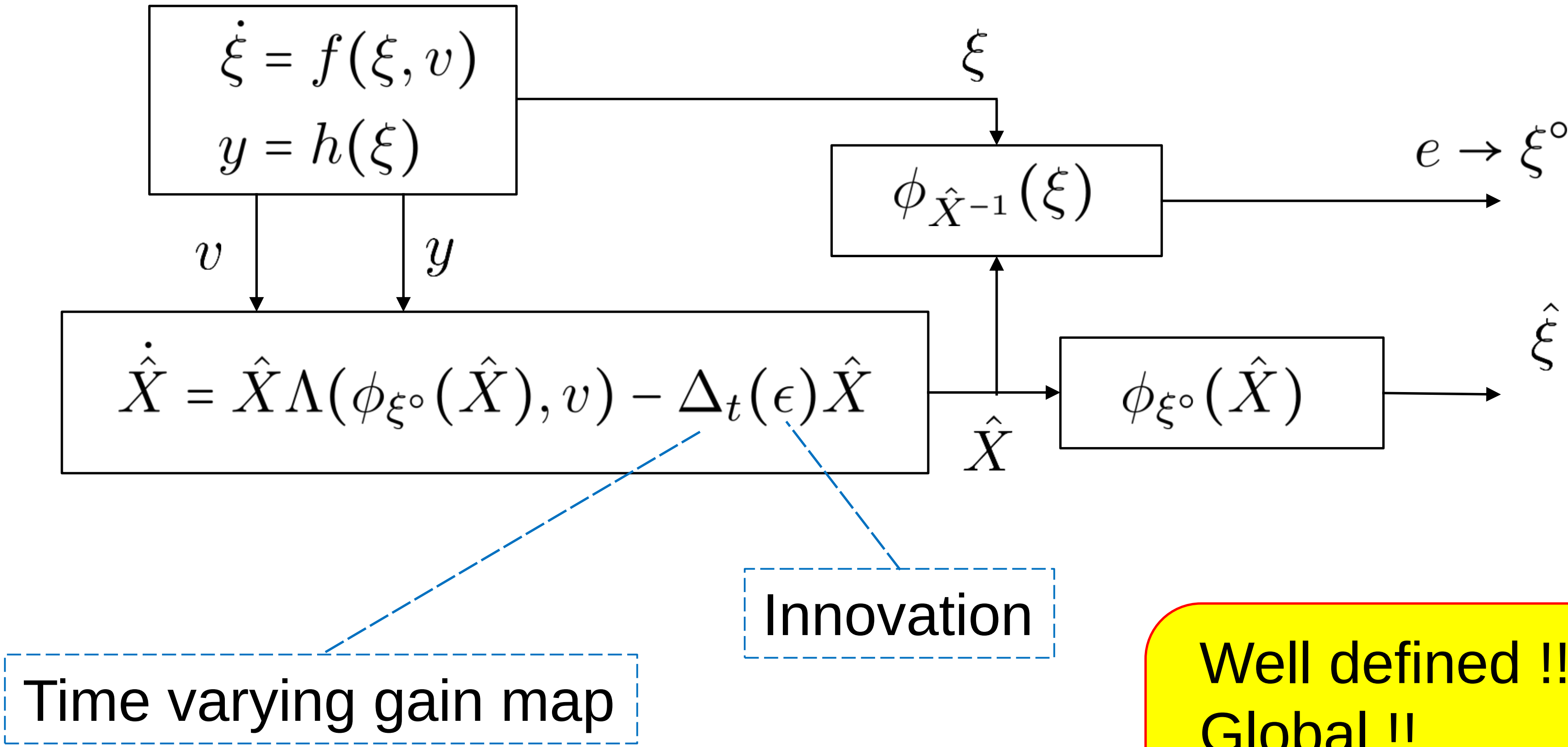
Equivariant output



Innovation


ϕ_{ξ°
ξ°
For equivariant output
$\epsilon = h(e)$
$\hat{X} \in \mathbf{G}$
State space
$\mathcal{M}$
$\epsilon = \rho_{\hat{X}^{-1}}(y)$
$h : \mathcal{M} \to \mathcal{N}$
Innovation
or output error
$y = h(\xi)$
Output space $\mathcal{N}$
$y^\circ = h(\xi^\circ)$

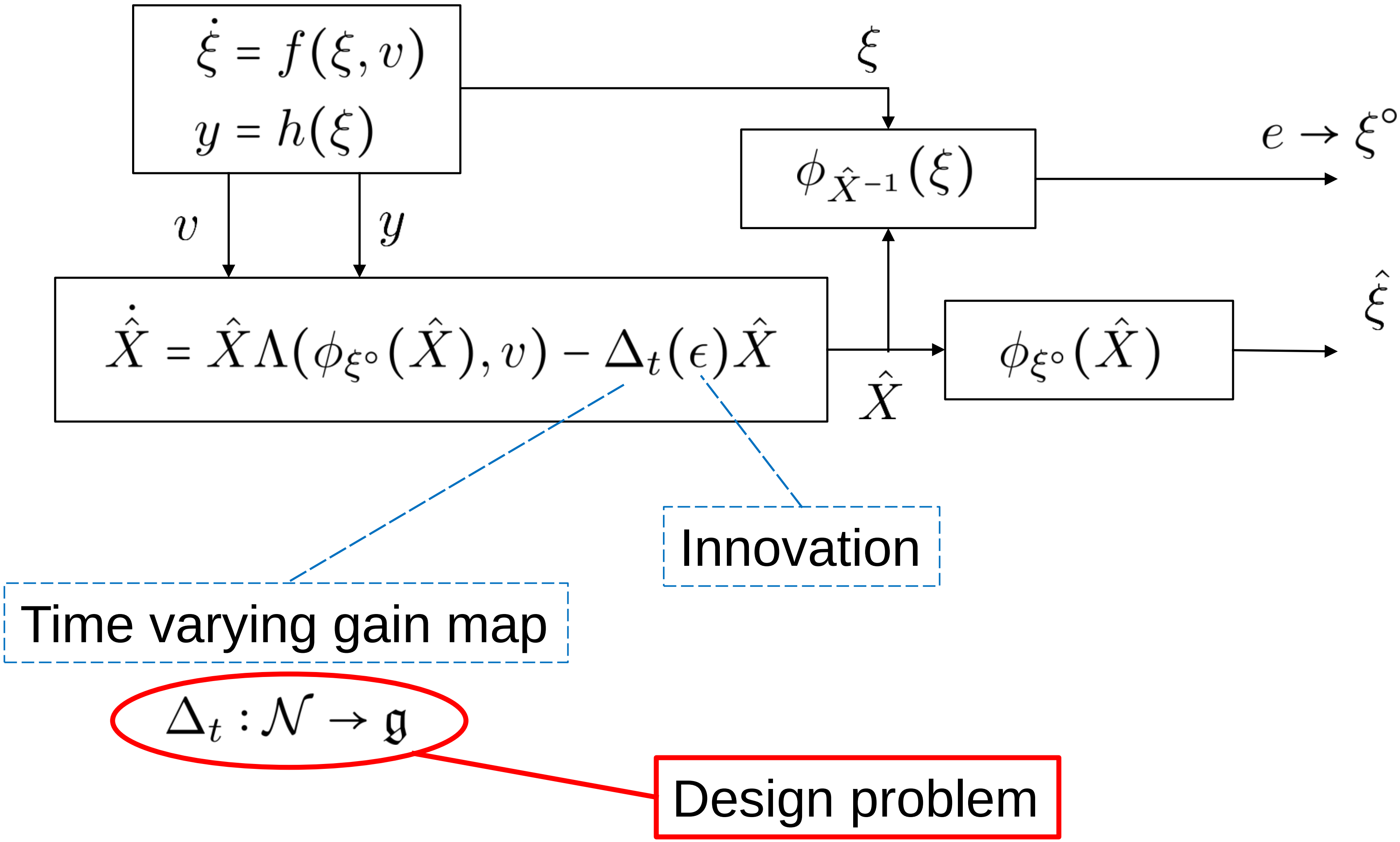
Equivariant observer: innovation



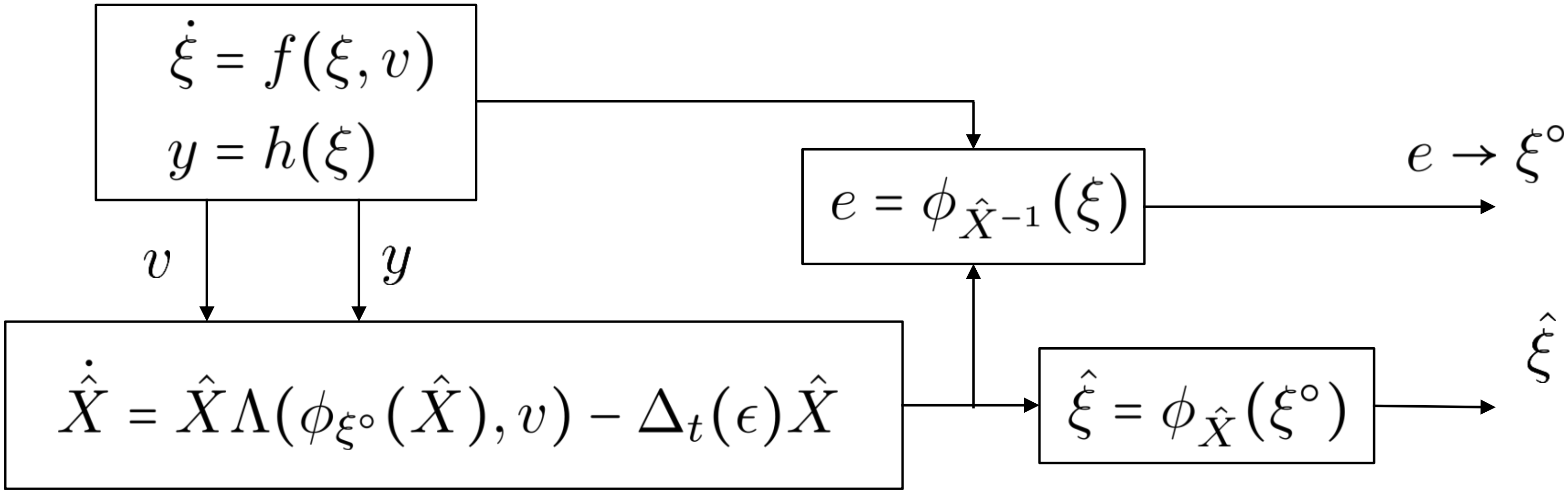
$$\Delta_t : \mathcal{N} \to \mathfrak{g}$$

Well defined !!
Global !!
Smooth !!
Computable !!

Equivariant observer: innovation



Error kinematics



$$\frac{\mathrm{d}}{\mathrm{d}t} e := \mathrm{d}\phi_e \, \mathrm{Ad}_{\hat{X}} \left(\Lambda(\phi_{\hat{X}}(\xi^\circ), v) - \Lambda(\xi, v) \right) - \mathrm{d}\phi_e \Delta_t(\epsilon)$$

Observer design remains a challenging problem due to non-autonomous error dynamics

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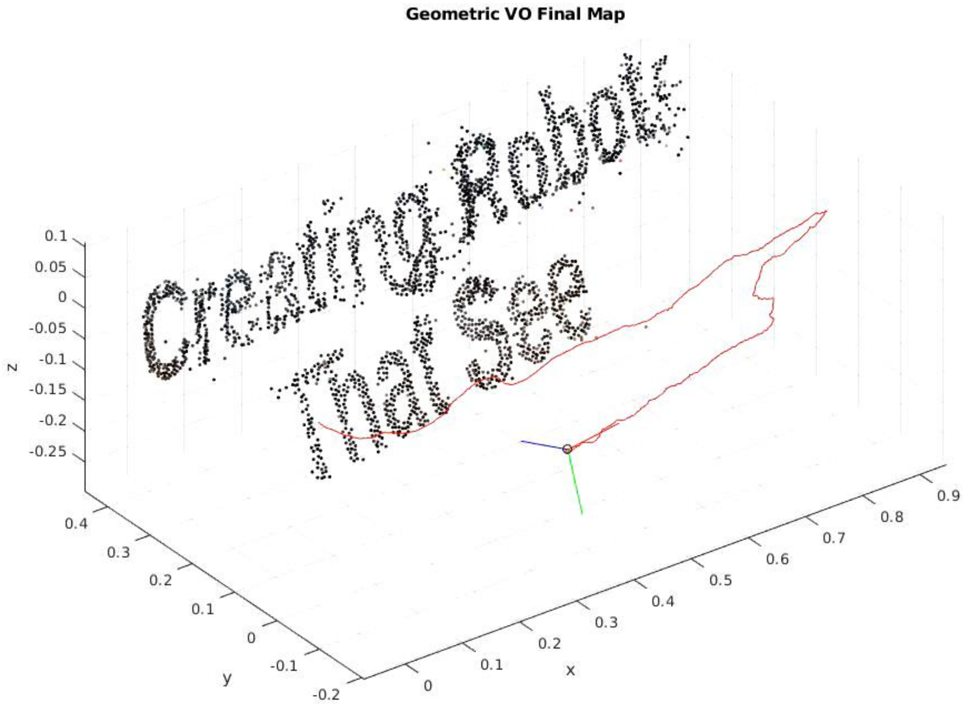
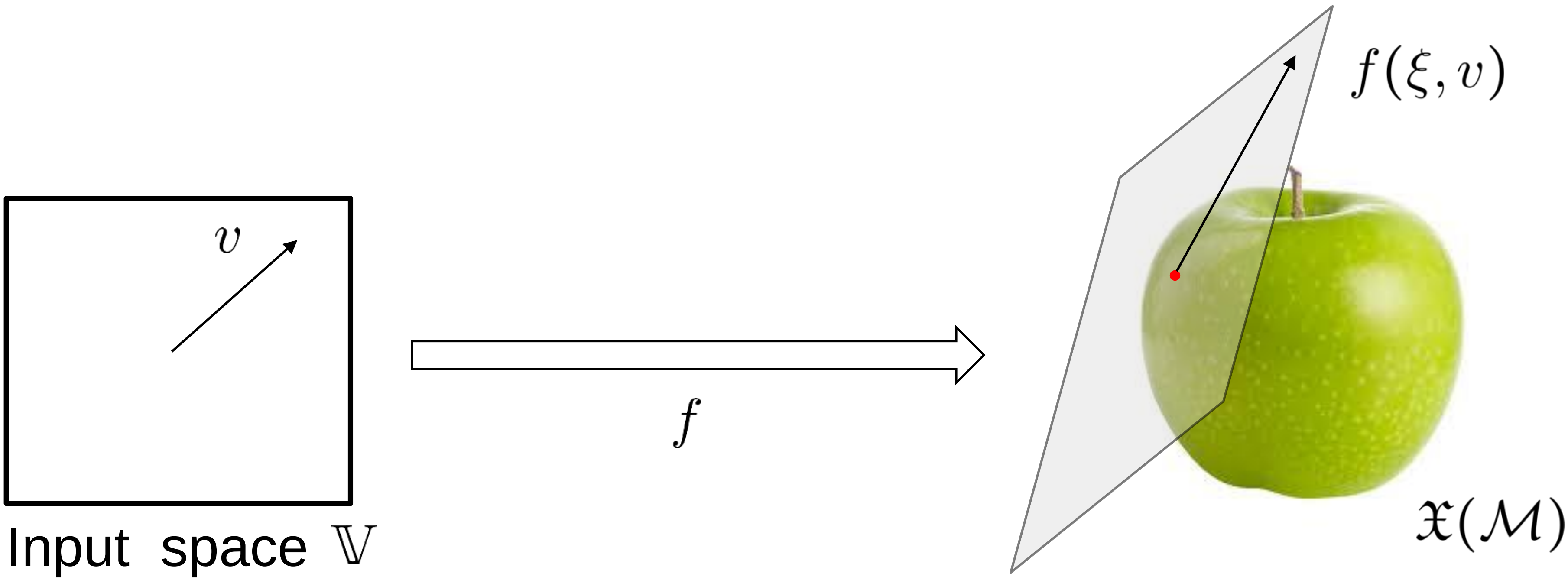
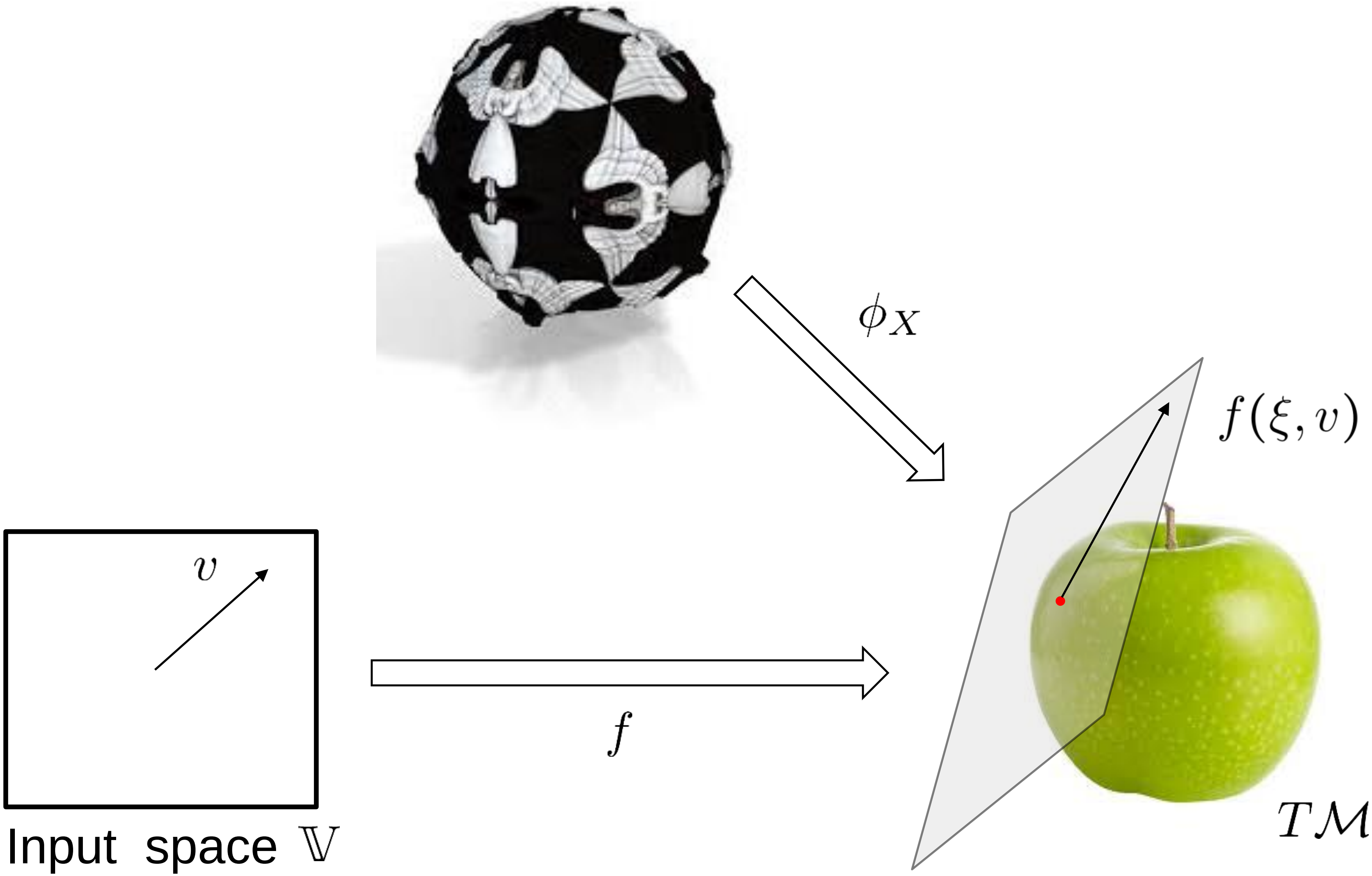


Input Symmetry and Equivariance


v
Input space $\mathbb{V}$
f
$f(\xi, v)$
$\mathfrak{X}(\mathcal{M})$

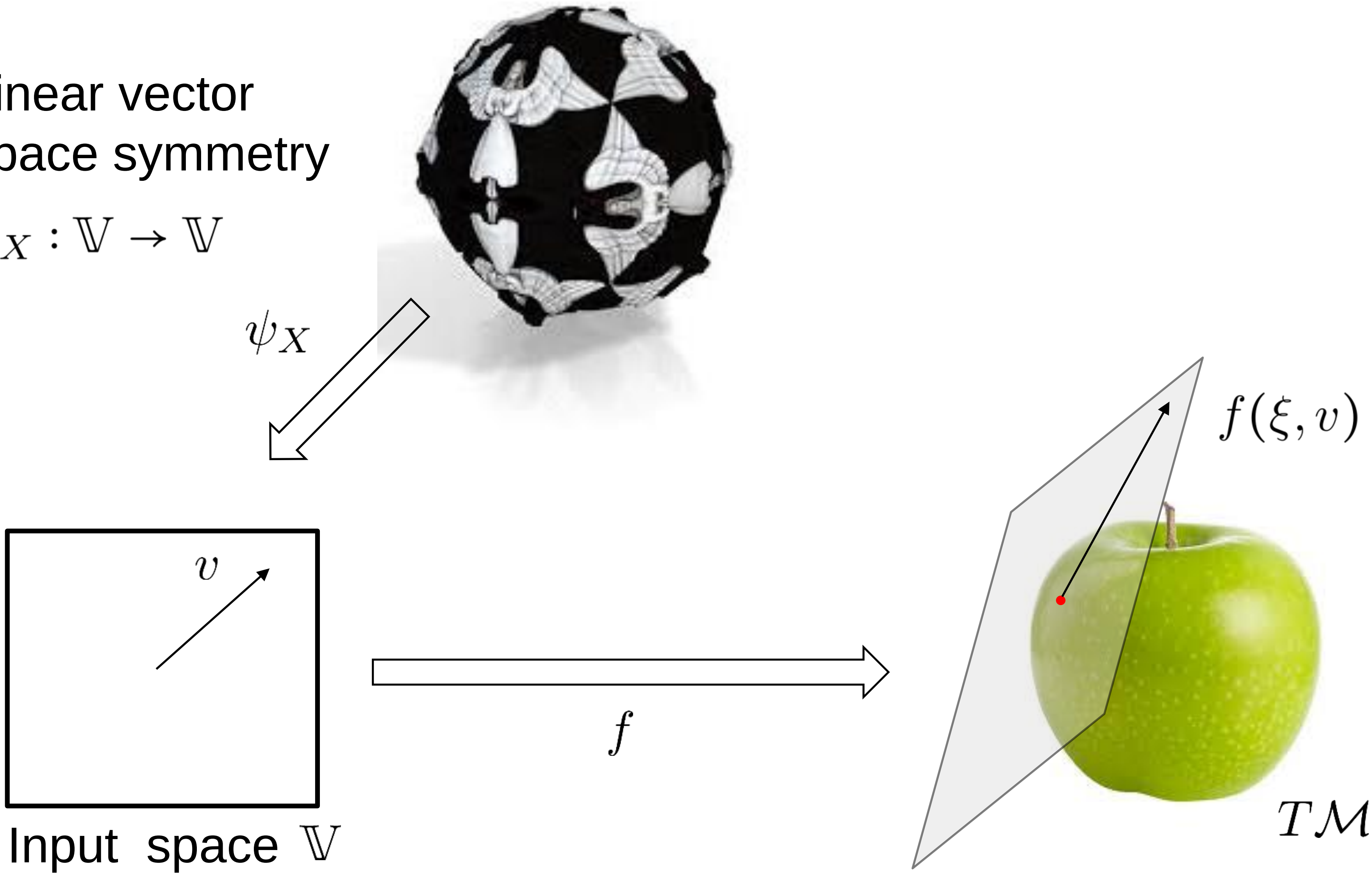
State Symmetry



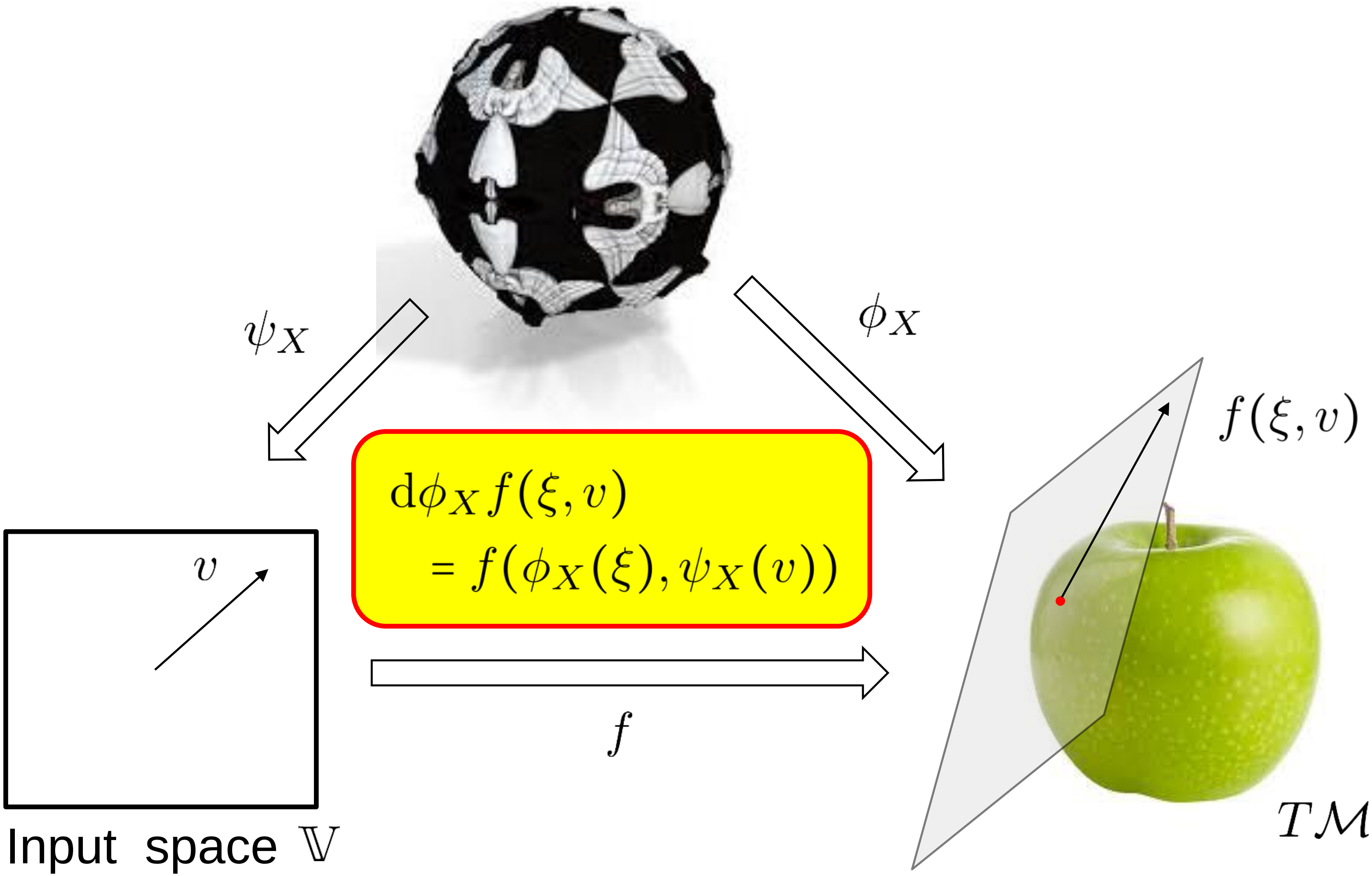
Input symmetry and equivariance

Linear vector space symmetry

$$\psi_X : \mathbb{V} \to \mathbb{V}$$



Input symmetry and equivariance



Intrinsic input symmetry

System function defines a family of smooth vector fields

$$f : \mathbb{V} \to \mathfrak{X}(\mathcal{M})$$
$$v \mapsto f_v$$

Smooth vector fields on $\mathcal{M}$

The input action is uniquely defined by the state action

$$f_{\psi_X(v)} := \mathrm{d}\phi_X \circ f_v \circ \phi_{X^{-1}} \in \mathfrak{X}(\mathcal{M})$$

$$f(\phi_X(\xi), \psi_X(v)) = f_{\psi_X(v)}(\phi_X(\xi)) = \mathrm{d}\phi_X f(\phi_{X^{-1}}(\phi_X(\xi)), v)$$
$$= \mathrm{d}\phi_X f(\xi, v)$$

The input space $\mathbb{V}$ can always be extended to make the system f equivariant.

Equivariant lifts and error kinematics

Definition: A lift $\Lambda : \mathcal{M} \times \mathbb{V} \to \mathfrak{g}$ is equivariant if

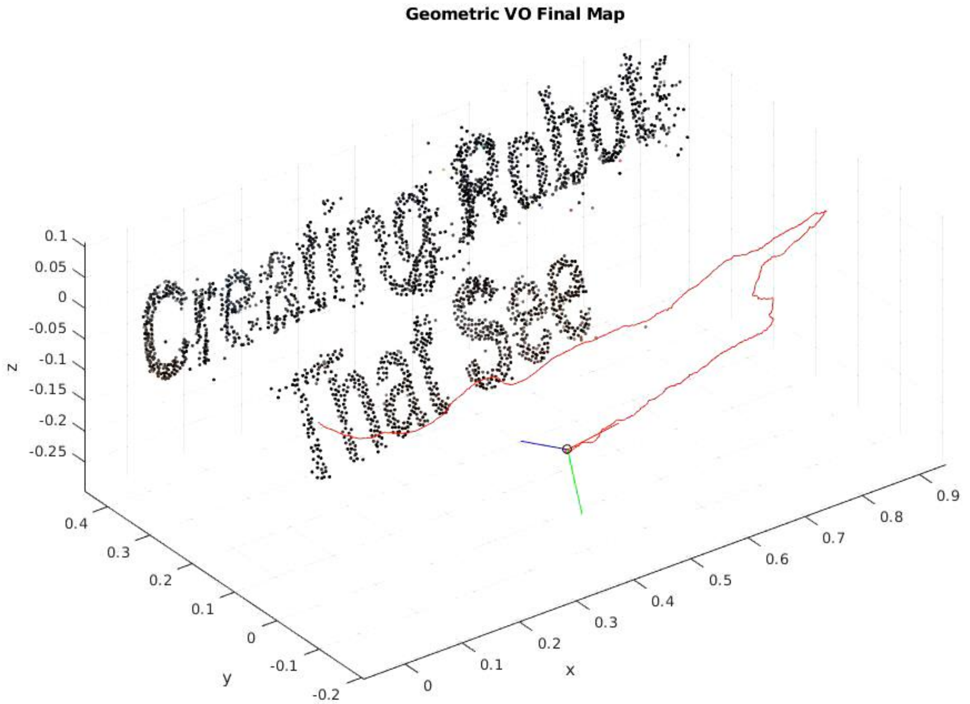
$$\mathrm{Ad}_{X^{-1}} \Lambda(\xi, v) = \Lambda(\phi_X(\xi), \psi_X(v))$$

Theorem: If a kinematic system is equivariant and the symmetry group $\mathbf{G}$ is reductive then an equivariant lift Λ exists.

$$\frac{\mathrm{d}}{\mathrm{d}t} e := \mathrm{d}\phi_e \, \mathrm{Ad}_{\hat{X}} \big(\Lambda(\phi_{\hat{X}}(\xi^\circ), v) - \Lambda(\xi, v) \big) - \mathrm{d}\phi_e \Delta_t(\epsilon)$$

⇩

$$\frac{\mathrm{d}}{\mathrm{d}t} e := \mathrm{d}\phi_e \big(\Lambda(\xi^\circ, \psi_{\hat{X}^{-1}}(v)) - \Lambda(e, \psi_{\hat{X}^{-1}}(v)) \big) - \mathrm{d}\phi_e \Delta_t(\epsilon)$$



Invariant Systems

Invariant systems

Definition: An equivariant lift $\Lambda : \mathcal{M} \times \mathbb{V} \to \mathfrak{g}$ is

Type I: if

$$\Lambda(\xi, v) = \Lambda(v)$$

Type II: if

$$\mathrm{Ad}_{X^{-1}} \Lambda(\xi, v) = \Lambda(\phi_X(\xi), v)$$

Equivalent to "group affine" Bonnabel *et al.*

Type I system kinematics

$$\dot{X} = X\Lambda(v)$$

Body-fixed velocity measurements

Type II system kinematics

$$\dot{X} = \Lambda(\xi^\circ, v)X$$

Reference-fixed velocity measurements

Invariant error kinematics

$$\frac{\mathrm{d}}{\mathrm{d}t} e := \mathrm{d}\phi_e \left(\Lambda(\xi^\circ, \psi_{\hat{X}^{-1}}(v)) - \Lambda(e, \psi_{\hat{X}^{-1}}(v)) \right) - \mathrm{d}\phi_e \Delta_t(\epsilon)$$

Type I system error kinematics

$$\frac{\mathrm{d}}{\mathrm{d}t} e = -k \mathrm{d}\phi_e \Delta_t(\epsilon)$$

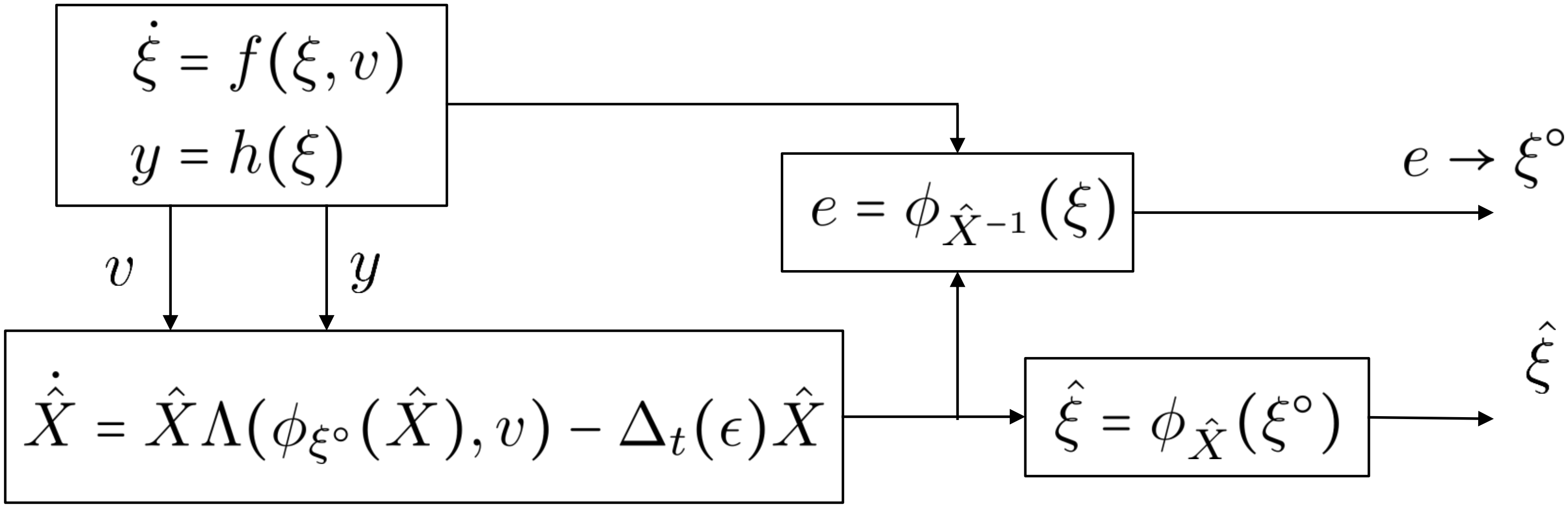
Autonomous error kinematics.

Type II system kinematics

$$\frac{\mathrm{d}}{\mathrm{d}t} e := \mathrm{d}\phi_e \left(\Lambda(\xi^\circ, v) - \Lambda(e, v) \right) - \mathrm{d}\phi_e \Delta_t(\epsilon)$$

Independent error kinematics.

Equivariant Observer Design

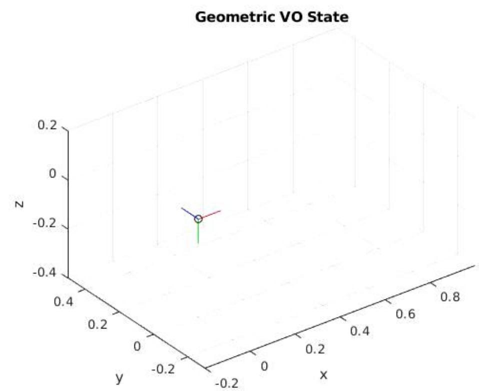


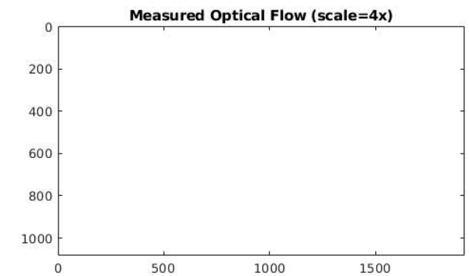
$$\frac{\mathrm{d}}{\mathrm{d}t} e := \mathrm{d}\phi_e \left(\Lambda(\xi^\circ, \psi_{\hat{X}^{-1}}(v)) - \Lambda(e, \psi_{\hat{X}^{-1}}(v)) \right) - \mathrm{d}\phi_e \Delta_t(\epsilon)$$

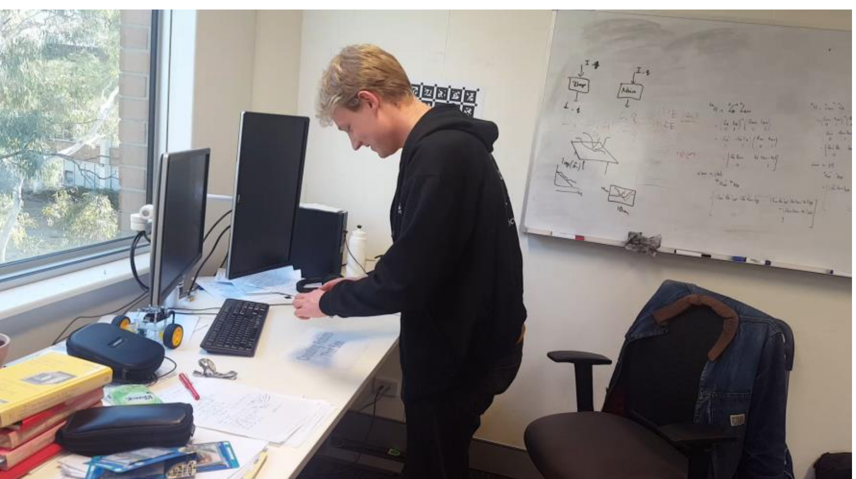
Design approaches:

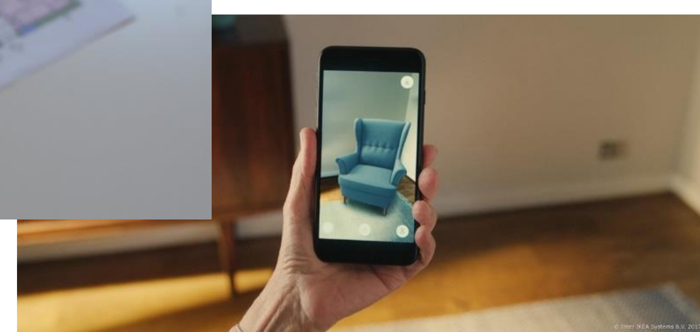
- Constructive nonlinear design for a Lyapunov function $\mathcal{L}(e)$.
- Linearise error kinematics around $e = \xi^\circ$ and use linear design.
- Minimum energy cost functional and approximation.

Spatial Awareness for Augmented Reality

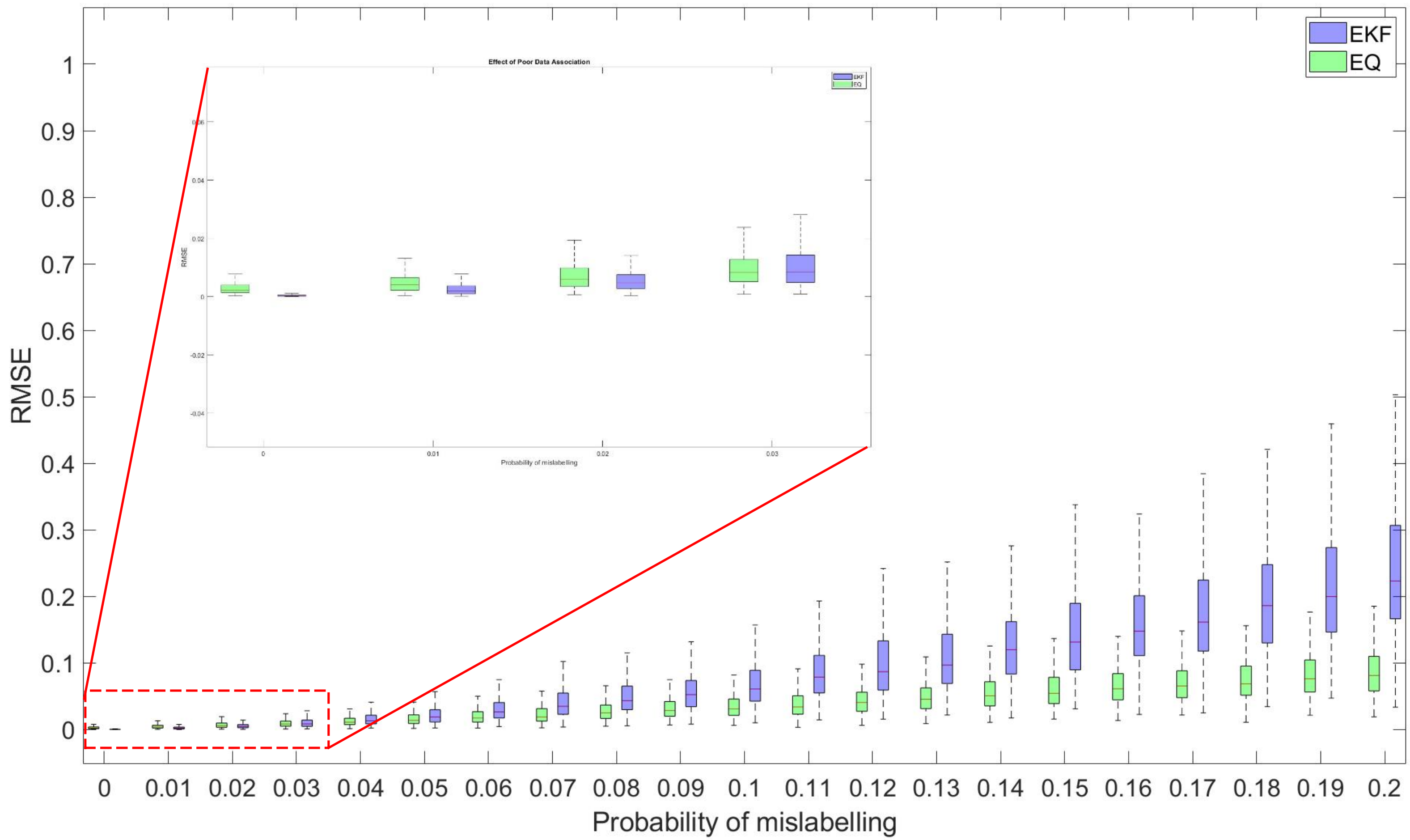








SLAM Robustness to data association error



Conclusions

Kinematics

Local error

$$\dot{\xi} = f(\xi, v)$$
$$y = h(\xi)$$

Conclusions

Kinematics

Local error

State Symmetry

Lifted system

Global error

$$\phi_{\hat{X}} : \mathcal{M} \to \mathcal{M}$$

$$\dot{\hat{X}} = \hat{X}\Lambda(\phi_{\xi^\circ}(\hat{X}), v)$$

$$e = \phi_{\hat{X}^{-1}}(\xi)$$

Conclusions

Kinematics

Local error

State Symmetry

Lifted system

Global error

Output Symmetry

Equivariant output

Global innovation

$$\rho_{\hat{X}} : \mathcal{N} \to \mathcal{N}$$

$$\epsilon = \rho_{\hat{X}^{-1}}(y)$$

$$\Delta_t(\epsilon)$$

Conclusions

Kinematics

Local error

State Symmetry

Lifted system

Global error

Output Symmetry

Equivariant output

Global innovation

Input Symmetry

Equivariant lift

Global error kinematics

$$\psi_{\hat{X}} : \mathbb{V} \to \mathbb{V}$$

…

$$\mathrm{Ad}_{X^{-1}} \Lambda(\xi, v) = \Lambda(\phi_X(\xi), \psi_X(v))$$

Conclusions

Kinematics

Local error

State Symmetry

Lifted system

Global error

Output Symmetry

Equivariant output

Global innovation

Input Symmetry

Equivariant lift

Global error kinematics

Invariance

Group affine

Independent error kinematics

(Bonnabel et al)

IEKF

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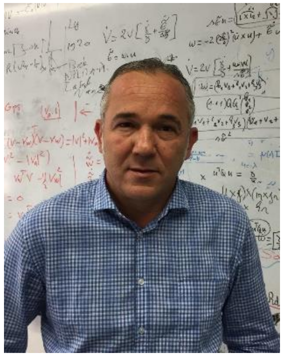

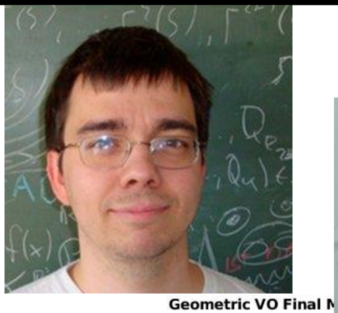

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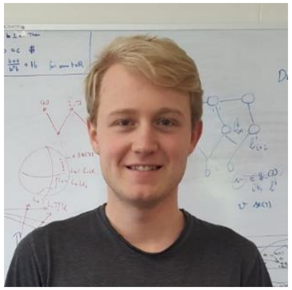

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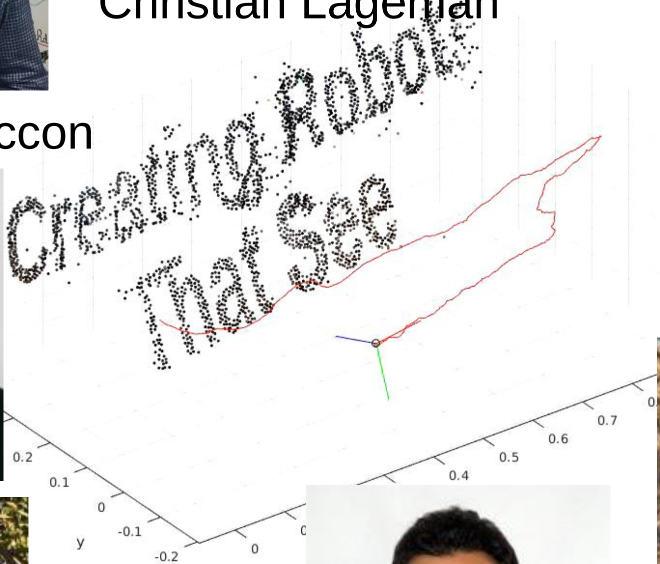

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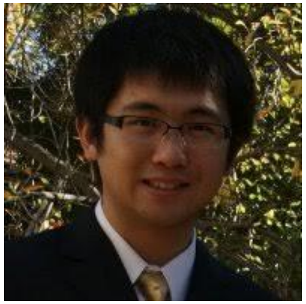

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