Distributed Machine Learning over Networks

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Scientific context

- **Proliferation of digital data**
  - Personal data
  - Industry
  - Scientific: from bioinformatics to humanities
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Outline

1. **Machine learning as optimization of finite sums**
   - Fast stochastic gradient methods for convex problems
   - "Optimal" single machine algorithms
2. **Distributed optimization over networks**
   - Gossip and accelerated gossip for decentralized optimization
   - "Optimal" algorithms for consensus optimization
3. **Distributed optimization for machine learning**
   - Finite sum of finite sums
   - Distribution of the fastest single machine algorithm

Parametric supervised machine learning

- **Data**: n observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \ldots, n$
- **Prediction function** $h(x, \theta) \in \mathbb{R}$ parameterized by $\theta \in \mathbb{R}^d$

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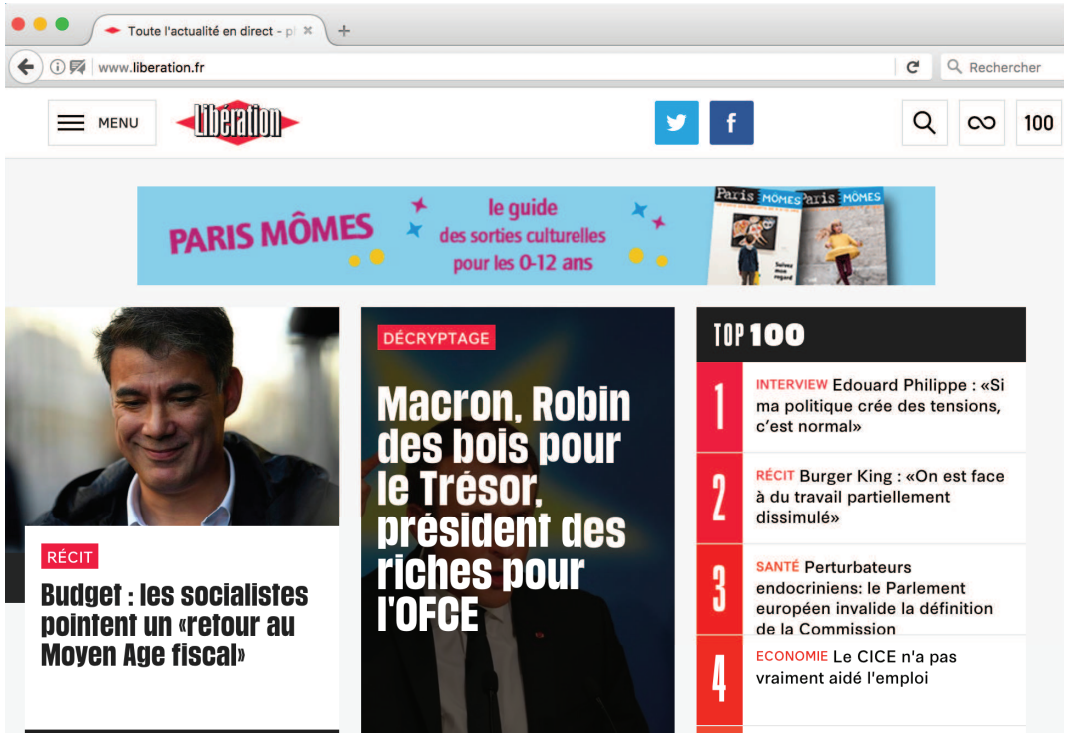

- **Advertising**: $n > 10^9$
  - $\Phi(x) \in \{0, 1\}^d$, $d > 10^9$
  - Navigation history + ad
- Linear predictions
  - $h(x, \theta) = \theta^\top \Phi(x)$

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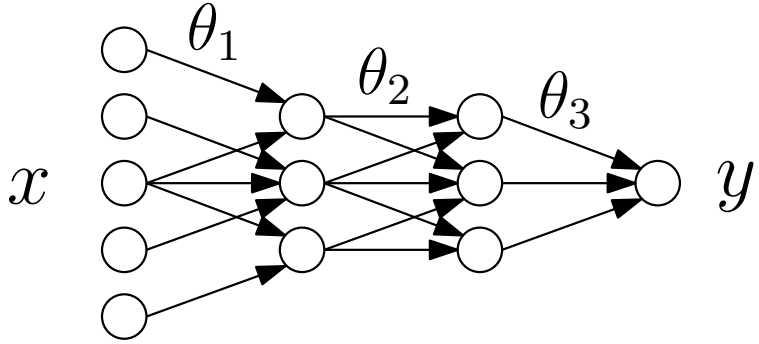
  - Neural networks $(n, d > 10^6)$: $h(x, \theta) = \theta_m^\top \sigma(\theta_{m-1}^\top \sigma(\cdots \theta_2^\top \sigma(\theta_1^\top x))$



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- **(regularized) empirical risk minimization**:

$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} \ell\big(y_i, h(x_i, \theta)\big) \quad + \quad \lambda \Omega(\theta)$$

data fitting term + regularizer

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(least-squares regression)

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(logistic regression)

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- **Machine learning through large-scale optimization**
  - Convex vs. non-convex optimization problems

Stochastic vs. deterministic methods

- Minimizing $g(\theta) = \frac{1}{n}\sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell\big(y_i, h(x_i, \theta)\big) + \lambda\Omega(\theta)$

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- Batch gradient descent: $\theta_t = \theta_{t-1} - \gamma \nabla g(\theta_{t-1}) = \theta_{t-1} - \frac{\gamma}{n}\sum_{i=1}^{n} \nabla f_i(\theta_{t-1})$
  - Exponential convergence rate in $O(e^{-t/\kappa})$ for convex problems
  - Can be accelerated to $O(e^{-t/\sqrt{\kappa}})$ (Nesterov, 1983)
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- Stochastic gradient descent: $\theta_t = \theta_{t-1} - \gamma_t \nabla f_{i(t)}(\theta_{t-1})$
  - Sampling with replacement: $i(t)$ random element of $\{1, \ldots, n\}$
  - Convergence rate in $O(\kappa/t)$
  - Iteration complexity is independent of n, typically $O(d)$

Recent progress in single machine optimization

- **Variance reduction**
  - Exponential convergence with $O(d)$ iteration cost
  - SAG (Le Roux, Schmidt, and Bach, 2012)
  - SVRG (Johnson and Zhang, 2013; Zhang et al., 2013)
  - SAGA (Defazio, Bach, and Lacoste-Julien, 2014), etc...

$$\theta_t = \theta_{t-1} - \gamma \left[\nabla f_{i(t)}(\theta_{t-1}) \right]$$

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$$\theta_t = \theta_{t-1} - \gamma \left[\nabla f_{i(t)}(\theta_{t-1}) + \frac{1}{n} \sum_{i=1}^{n} z_i^{t-1} - z_{i(t)}^{t-1} \right]$$

(with z_i^t stored value at time t of gradient of the i-th function)

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- **Running-time to reach precision** ε (with $\kappa =$ condition number)

Stochastic gradient descent	$d \times \kappa \times \frac{1}{\varepsilon}$
Gradient descent	$d \times n\kappa \times \log \frac{1}{\varepsilon}$
Variance reduction	$d \times (n + \kappa) \times \log \frac{1}{\varepsilon}$

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Variance reduction	$d \times (n + \kappa) \times \log \frac{1}{\varepsilon}$

  - Can be accelerated (e.g., Lan, 2015): $n + \kappa \Rightarrow n + \sqrt{n\kappa}$
  - Matching upper and lower bounds of complexity

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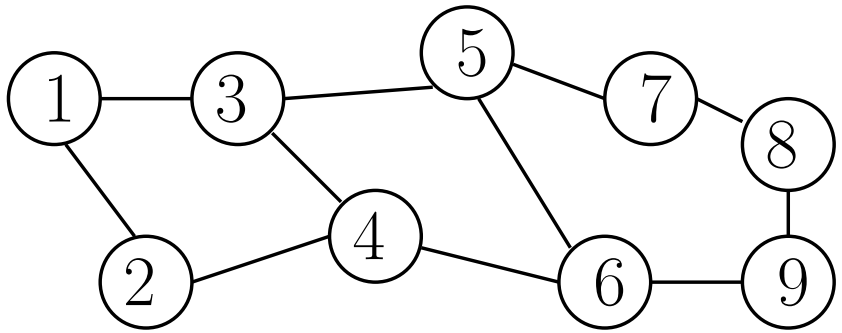
- **Consensus optimization**

$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) = g(\theta)$$

- **Each function f_i only accessible by node i in a graph**



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  - Example: $f_i(\theta)$ error of model defined by θ on dataset indexed by i

$$f_i(\theta) = \frac{1}{m_i} \sum_{j=1}^{m_i} \ell(y_{ij}, \theta^\top \Phi(x_{ij})) \text{ if } m_i \text{ observations}$$

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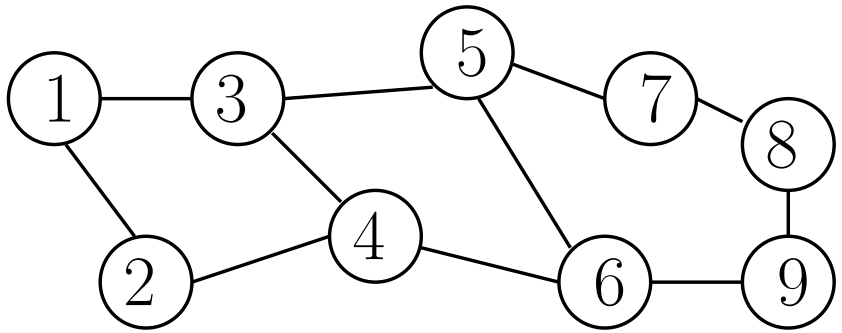

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- **Each function f_i only accessible by node i in a graph**
  - Massive datasets, multiple machines / cores
  - Communication / legal constraints

- **Goal: Minimize communication and local computation time**

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- **Consensus optimization**

$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) = g(\theta)$$

- **Why not simply distributing a fast single machine algorithm?**
  - (accelerated) gradient descent (see, e.g., Nesterov, 2004)

$$\theta_t = \theta_{t-1} - \gamma \nabla g(\theta_{t-1})$$

  - Requires $\sqrt{\kappa} \log \frac{1}{\varepsilon}$ full gradient computations to reach precision ε
  - Need to perform distributed averaging over a network

Classical algorithms for distributed averaging

- **Goal**: Given n observations $\xi_1, \ldots, \xi_n \in \mathbb{R}$
  - Compute $\theta_* = \frac{1}{n}\sum_{i=1}^{n} \xi_i = \arg\min_{\theta \in \mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} (\theta - \xi_i)^2$

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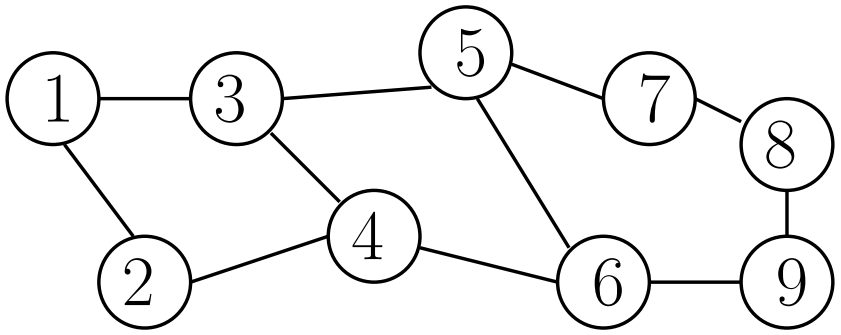
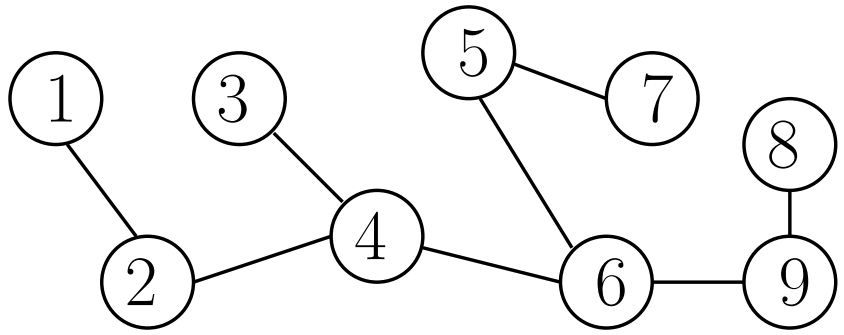
- **Centralized algorithms**
  - Compute a spanning tree with diameter $\leqslant 2\Delta$
  - Master/slave algorithm





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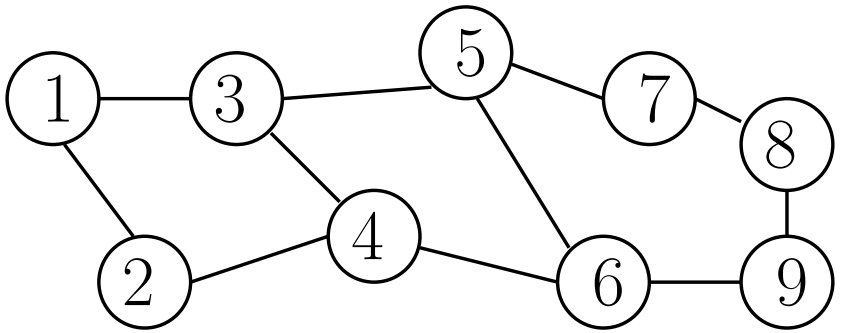

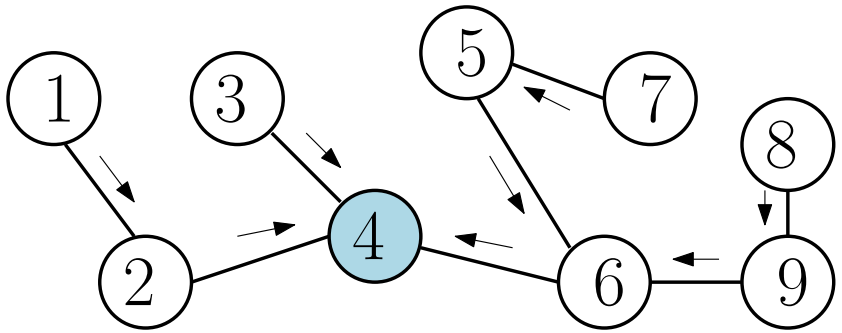

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- **Application to centralized distributed optimization**
  - $\sqrt{\kappa} \log \frac{1}{\varepsilon}$ gradient steps and $\sqrt{\kappa}\Delta \log \frac{1}{\varepsilon}$ communication steps
  - "Optimal" (Scaman, Bach, Bubeck, Lee, and Massoulié, 2017)

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- **Robustness?**

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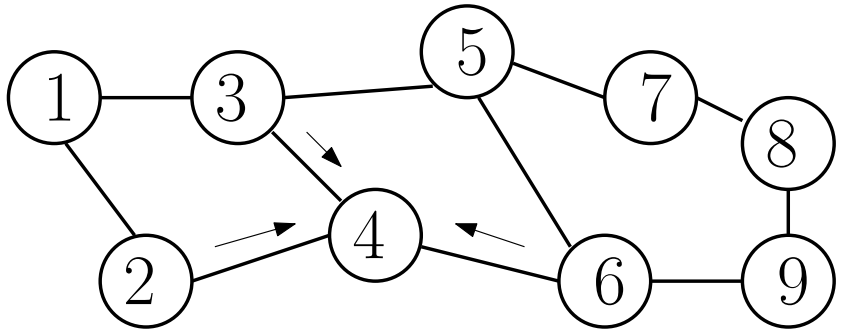

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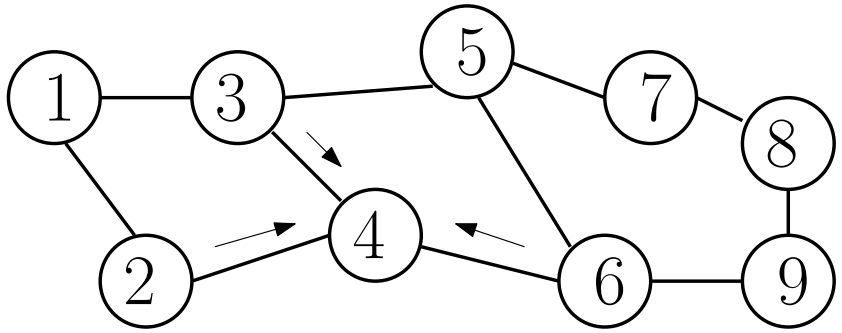

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  - Main iteration: $\boxed{\theta^{(t)} = W\theta^{(t-1)} = W^t\theta^{(0)} = W^t\xi}$
  - Typical assumption: W symmetric doubly stochastic matrix

Convergence of synchronous gossip

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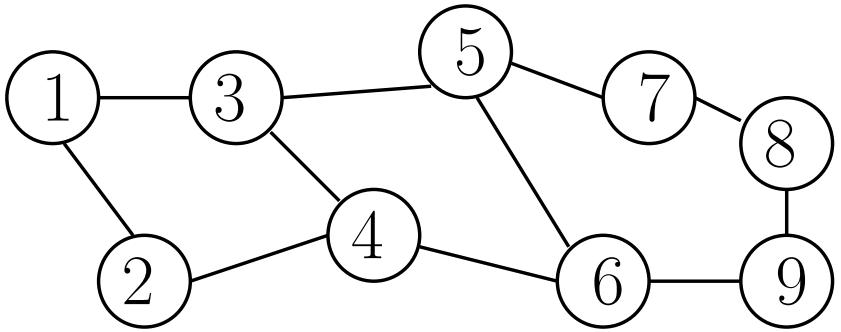
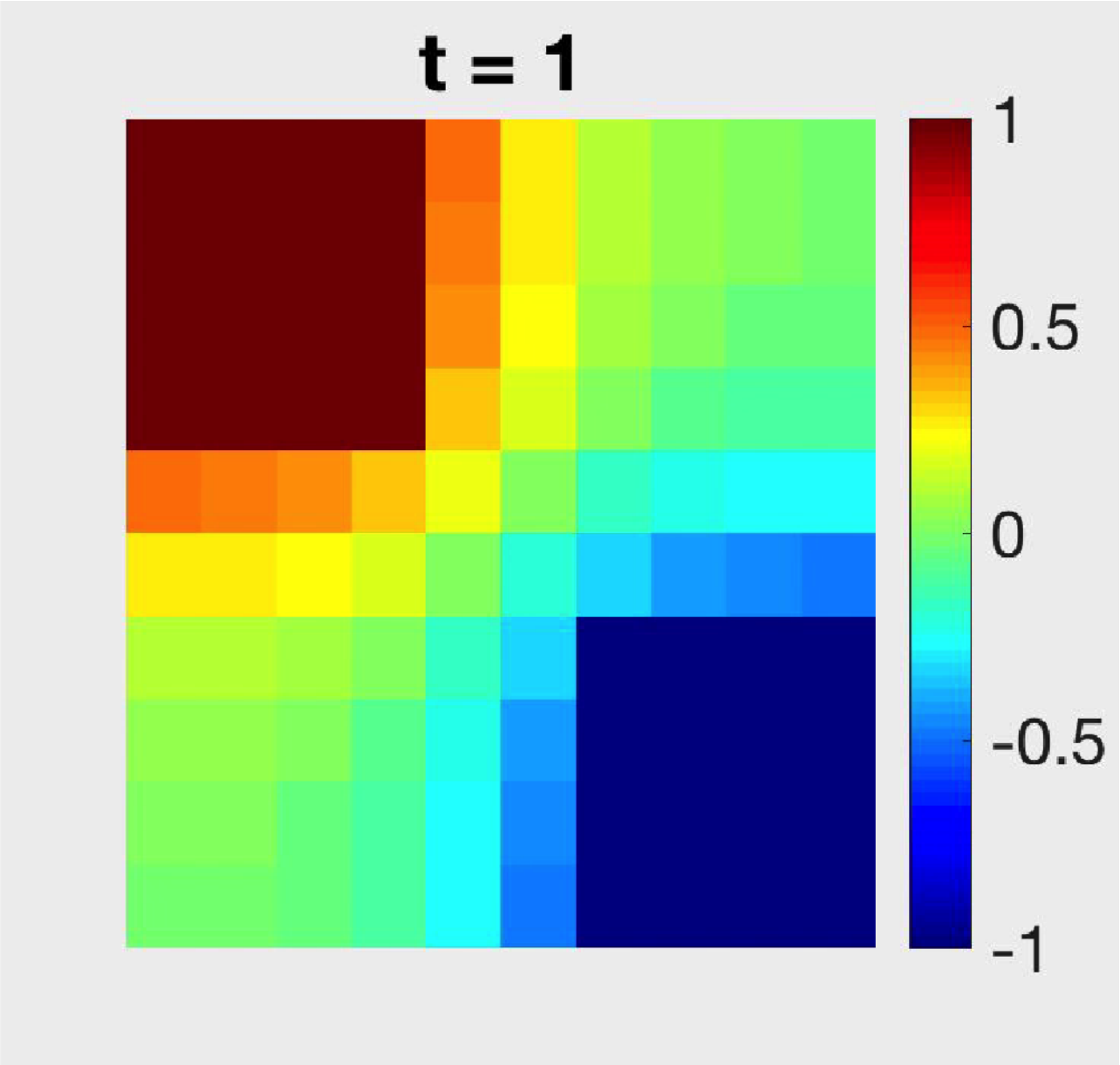
  - Typical assumption: W symmetric doubly stochastic matrix
  - Consequence: eigenvalues$(W) \in [-1, 1]$
  - Eigengap $\gamma = \lambda_1(W) - \lambda_2(W) = 1 - \lambda_2(W)$
  - $\gamma^{-1} =$ mixing time of the associated Markov chain



  - Need $\frac{1}{\gamma}\log\frac{1}{\varepsilon}$ iterations to reach precision ε

Illustration of synchronous gossip

- Two-dimensional grid



Decentralized optimization

- **Mixing gossip and optimization**
  - Nedic and Ozdaglar (2009)
  - Duchi et al. (2012); Wei and Ozdaglar (2012); Iutzeler et al. (2013); Shi et al. (2015); Jakovetić et al. (2015); Nedich et al. (2016); Mokhtari et al. (2016); Colin et al. (2016); Scaman et al. (2017), **etc.**

  - Mostly for convex problems

Decentralized optimization

- **Mixing gossip and optimization**

- **Lower bound on complexity** (Scaman et al., 2017)
  - $\sqrt{\kappa}\log\frac{1}{\varepsilon}$ gradient steps and $\sqrt{\kappa/\gamma}\log\frac{1}{\varepsilon}$ communication steps
  - Plain gossip not optimal!
    (need to gossip gradients with increasing precision)

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- **Is this lower bound achievable?**

Dual reformulation (Jakovetić et al., 2015)

$$\min_{\theta \in \mathbb{R}^d} \sum_{i=1}^n f_i(\theta) = \min_{\theta^{(1)}, \ldots, \theta^{(n)} \in \mathbb{R}^d} \sum_{i=1}^n f_i(\theta^{(i)}) \text{ such that } \forall i \sim j, \theta^{(i)} = \theta^{(j)}$$

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- **Accelerated gradient ascent** (Scaman et al., 2017)
  $\Leftrightarrow$ alternating local gradient computations and a gossip step
  - $\sqrt{\kappa/\gamma}\log\frac{1}{\varepsilon}$ gradient steps and $\sqrt{\kappa/\gamma}\log\frac{1}{\varepsilon}$ communication steps
  - Not optimal $\Rightarrow$ need accelerated gossip

Accelerated gossip

- **Regular gossip**
  - Iterations: $\theta^{(t)} = W^t \theta^{(0)}$
- **Accelerated gossip**
  - Chebyshev acceleration (Auzinger, 2011; Arioli and Scott, 2014)
  - Shift-register gossip (Cao et al., 2006)
  - Linear combinations $\Leftrightarrow \eta^{(t)} = \sum_{k=0}^{t} \alpha_k \theta^{(k)} = \sum_{k=0}^{t} \alpha_k W^k \xi = P_t(W)\xi$

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  - Optimal polynomial is the Chebyshev polynomial
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$$\theta^{(t)} = \omega_t W \theta^{(t-1)} + (1 - \omega_t)\theta^{(t-2)}$$

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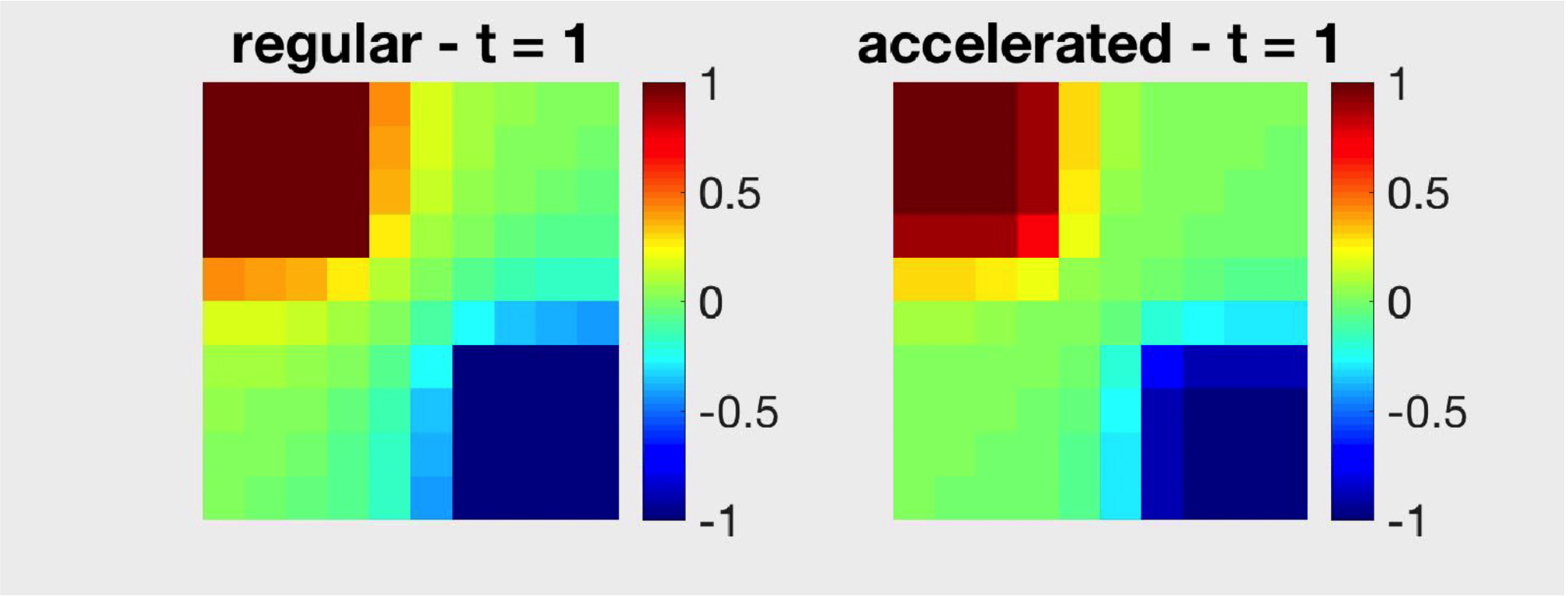
  - Replace γ^{-1} by $\gamma^{-1/2}$ in rates

Illustration of accelerated gossip

- Two-dimensional grid



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$\Rightarrow$ **Optimal complexity for optimization** (Scaman et al., 2017)

Distributed machine learning over networks

Outline

1. **Machine learning as optimization of finite sums**
   - Fast stochastic gradient methods for convex problems
   - "Optimal" single machine algorithms
2. **Distributed optimization over networks**
   - Gossip and accelerated gossip for decentralized optimization
   - "Optimal" algorithms for consensus optimization
3. **Distributed optimization for machine learning**
   - Finite sum of finite sums
   - Distribution of the fastest single machine algorithm

Distributed machine learning over networks

- **Consensus optimization**

$$\min_{\theta \in \mathbb{R}^d} \quad \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) = g(\theta)$$

  - $f_i(\theta)$ error of model defined by θ on dataset indexed by i
  - Example: $f_i(\theta) = \frac{1}{m_i} \sum_{j=1}^{m_i} \ell(y_{ij}, \theta^\top \Phi(x_{ij}))$ if m_i observations

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- **Single machine vs. "optimal" decentralized algorithm**

Algorithm	gradient steps	communication
Single machine algorithm	$nm + \sqrt{nm\kappa}$	0
MSDA (Scaman et al., 2017)	$m\sqrt{\kappa}$	$\sqrt{\kappa/\gamma}$

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- **MSDA (Scaman et al., 2017)**

  - $\sqrt{\kappa} \log \frac{1}{\varepsilon}$ gradient steps and $\sqrt{\kappa/\gamma} \log \frac{1}{\varepsilon}$ communication steps
  - "Optimal", but still not adapted to machine learning
  - Huge slow down when going from 1 to 2 machines
  - Only synchronous

Decentralized algorithms for machine learning (Hendrikx, Bach, and Massoulié, 2019)

- **Trade-offs between gradient and communication steps**
  - Adapted to functions of the type $f_i(\theta) = \frac{1}{m}\sum_{j=1}^{m} \ell(y_{ij}, \theta^\top \Phi(x_{ij}))$
  - Allows for partial asynchrony
- n **computing nodes, with** m **observations each**

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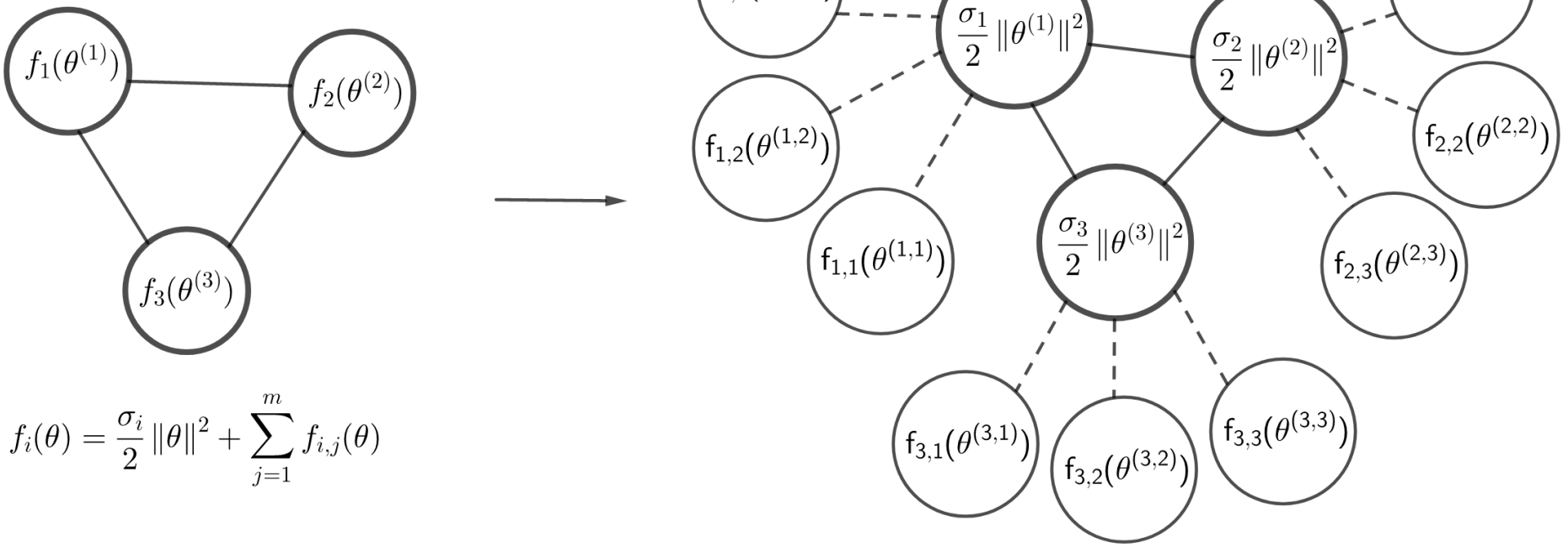
ADFS (Hendrikx et al., 2019)	$m + \sqrt{m\kappa}$	$\sqrt{\kappa/\gamma}$

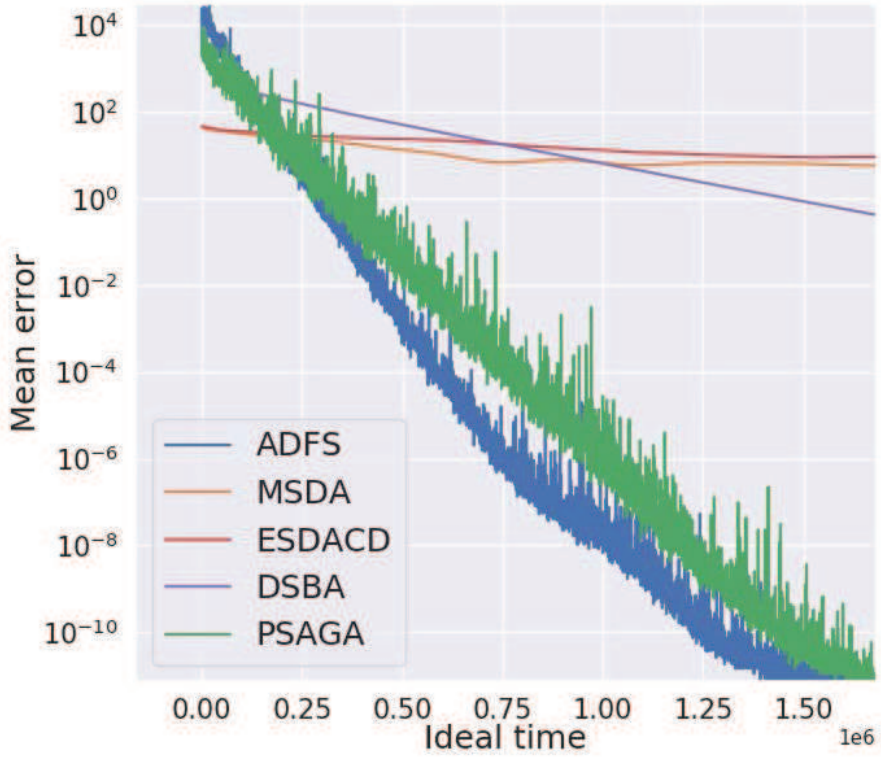
ADFS - Algorithm principle

- **Minimizing** $\sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} f_{i,j}(\theta) + \frac{\sigma_i}{2} \|\theta\|^2 \right\}$
  - Create an equivalent graph
  - Dual randomized coordinate ascent (with non uniform sampling)
  - Decoupling of data and gossip steps

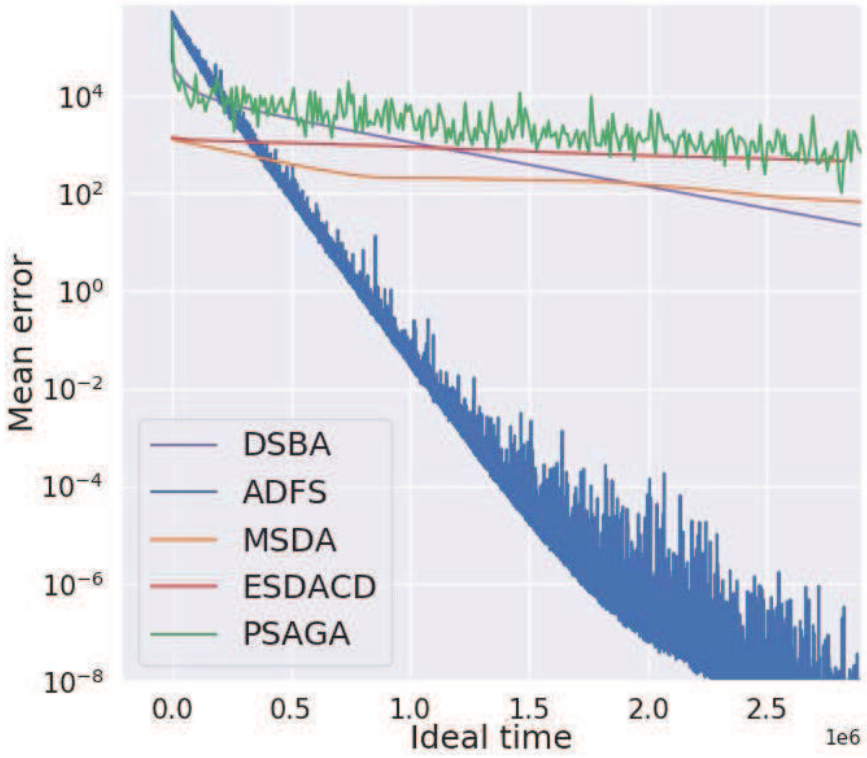


Decentralized algorithms for machine learning (Hendrikx, Bach, and Massoulié, 2019)

- **Running times on an actual cluster**
  - Logistic regression with $m = 10^4$ observations per node in $\mathbb{R}^{28}$
  - Two-dimensional grid network



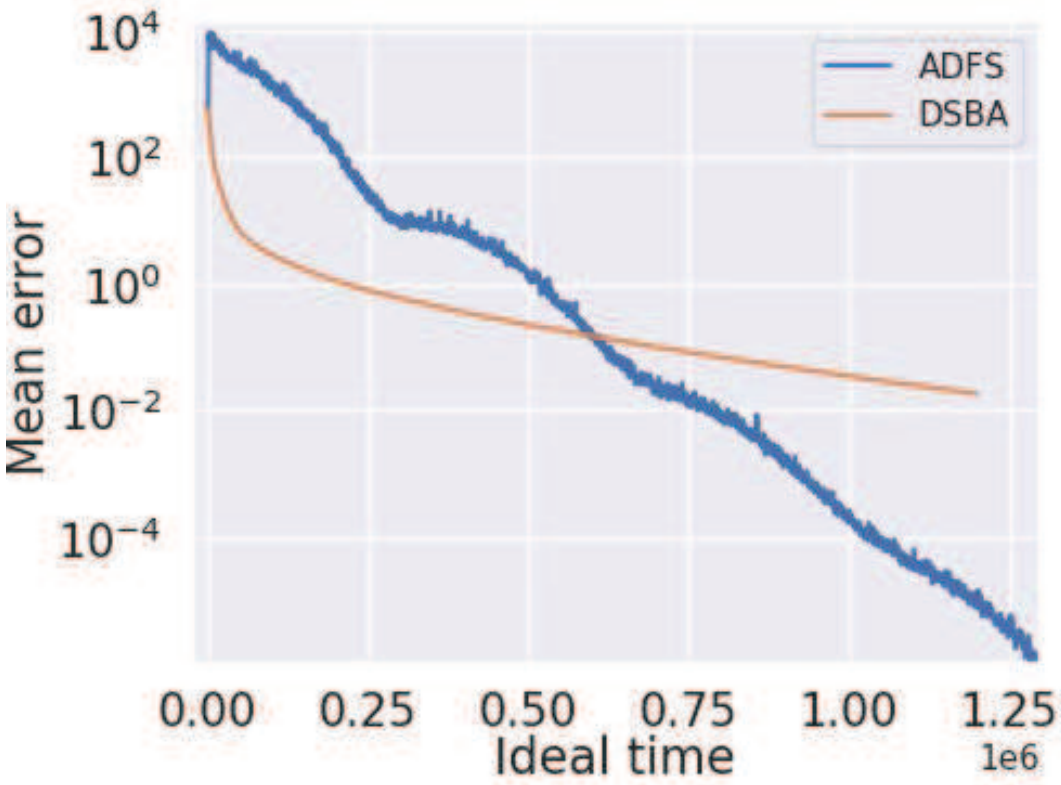
$n = 4$



$n = 100$

Decentralized algorithms for machine learning (Hendrikx, Bach, and Massoulié, 2019)

- **Running times on an actual cluster**
  - Logistic regression with $mn \approx 10^5$ observations in $\mathbb{R}^{47\,236}$
  - Two-dimensional grid network with $n = 100$ nodes



Conclusions

- **Distributed decentralized machine learning**
  - Distributing the fastest single machine algorithms!
  - n machines and m observations per machine
  - From $nm + \sqrt{nm\kappa}$ (single machine) to $m + \sqrt{m\kappa}$ gradient steps
  - Linear speed-ups for well-conditioned problems

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  - Full asynchrony
  - Beyond convex problems
  - Matching running time complexity lower bounds
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- **Acceleration beyond optimization?**

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