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The curse of linearity and time-invariance

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Motivations

A semi-plenary at the CDC provides a unique opportunity to reflect on the past and to look at the future …

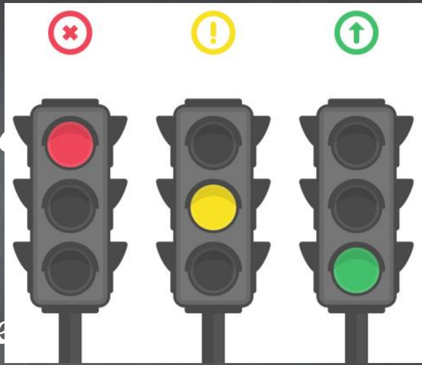
I spent 20+ years understanding linear objects from a nonlinear perspective and I now believe that

linearity and time-invariance are a curse

*They confuse our intuition and delay our understanding ….
pretty much like Euclidean geometry and the standard notion of orthogonality obfuscate our understanding of space*

Before spending another 20+ years in trying to (mis-)understand more linear objects I would like to reflect on some of the lessons I have learnt.

Motivations



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The rules

How to break the curse of linearity and time-invariance ✓

1. Study linear, time-invariant systems
2. Avoid matrix multiplication/inversion
3. While differentiating, keep track of constant terms
4. Never use frequency or Laplace transforms
5. If need use, matrices always act on something
6. Replace linear algebra with interconnection, invariance, pde's, coordinates transformations, the principle of optimality, dynamic programming, trajectories, differential operators, graph theory

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The plan

1. *Moments and phasors (Giordano Scarciotti)*
2. *The Loewner functions (Joel Simard)*
3. *Persistence of excitation (Alberto Padoan)*
4. *Adaptive control (Kaiwen Chen)*
5. *Optimal control (Mario Sassano)*

Items 1–3: **Analysis**

Items 4–5: **Design**

Moments are ubiquitous in maths/physics/biology ....

$$\eta_m = \int_? r^m \text{density}(r) dr \qquad r = \text{distance}$$

Moments in probability: 1, mean, variance, skewness, ...

Moment of a force/torque: first order moment

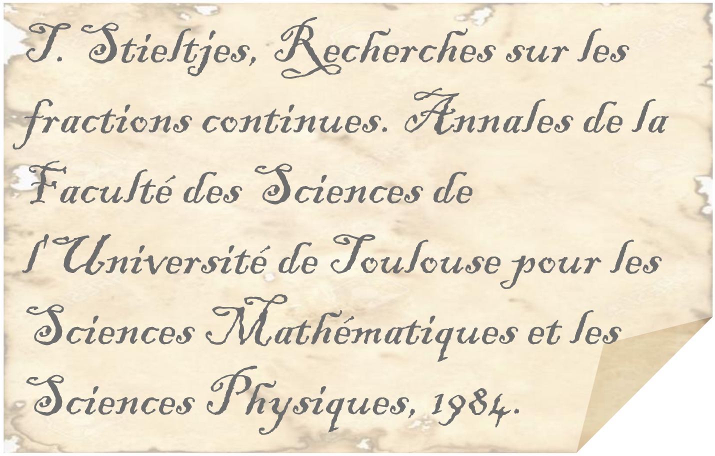
Electrical dipole: first order moment

Moment of inertia: second order moment

Phasors: first order moments

Biomass concentration: first order moment

Cell density: zero-th order moment


J. Stieltjes, Recherches sur les fractions continues. Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques, 1984.

... and systems theory

$$h : t \to I\!R \qquad \eta_m(s^\star) = \int_0^\infty t^m e^{-s^\star t} h(t) dt$$

h is the impulse response of a linear time-invariant (SISO) system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad W(s) = C(sI - A)^{-1}B$$

0-moment at $s^\star$: $\eta_0(s^\star) = C(s^\star I - A)^{-1}B$

k-moment at $s^\star$:

$$\eta_k(s^\star) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}\left(C(sI-A)^{-1}B\right)\right]_{s=s^\star} = C(s^\star I - A)^{-(k+1)}B$$

Moments: from transfer functions to state space

$$h : t \to \mathbb{R} \qquad \eta_m(s^\star) = \int_0^\infty t^m e^{-s^\star t} h(t)\,dt$$

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$$\eta_0(s^\star) = C\Pi$$

$$A\Pi + B = \Pi s^\star$$

Moments: from transfer functions to state space

$$h : t \to I\!R \qquad \eta_m(s^\star) = \int_0^\infty t^m e^{-s^\star t} h(t)\,dt$$

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$$\eta_0(s^\star), \ldots, \eta_k(s^\star) \iff \begin{cases} C\Pi \\ A\Pi + BL = \Pi S \\ \det(sI - S) = (s - s^\star)^{k+1} \end{cases}$$

The notion of moment – Linear systems

The interpolation point

$$\dot{\omega} = S\omega$$

The system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\dot{\omega} = S\omega \quad \xrightarrow{u = L\omega} \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{y}$$

Steady state response

Moments ✓

Π

$Re(\sigma(S)) \geq 0$

Observability

Excitability

Asymptotic stability

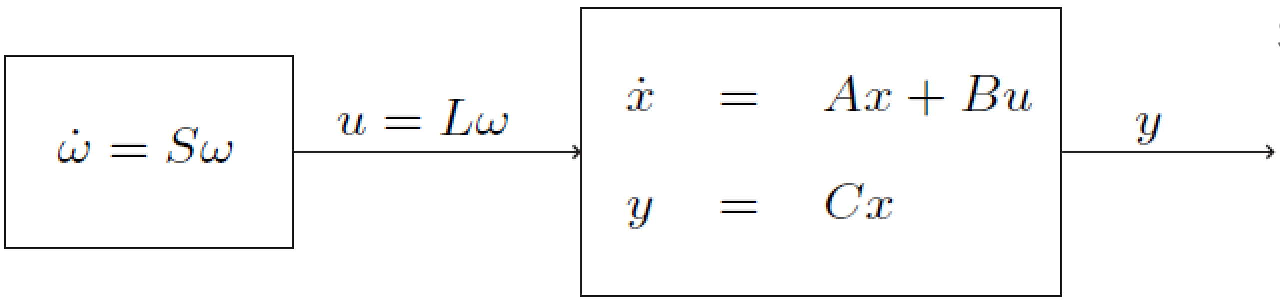
The notion of moment – Linear systems

The interpolation point

$$\dot{\omega} = S\omega$$

The system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$



Steady state response

Moments

Π

Alternatively

$$\dot{\omega} = S\omega + \delta_0\, \omega(0)$$

$$\omega(0) = 0$$

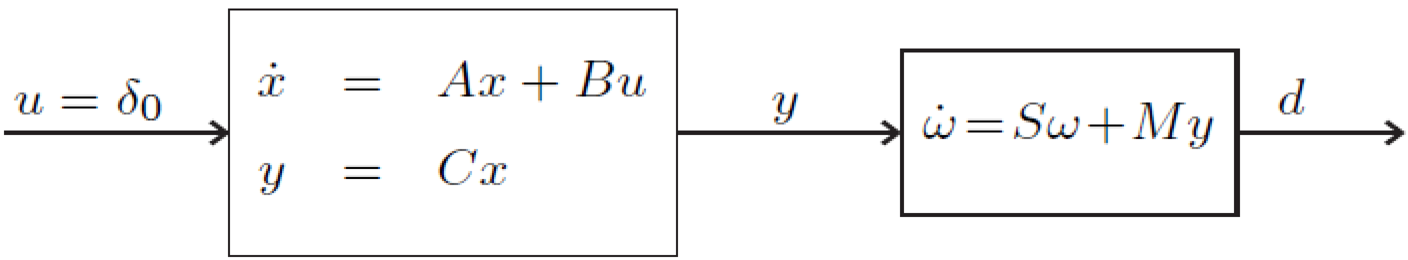
The notion of moment – Linear systems – *Swapped*

The interpolation point

$$\dot{\omega} = S\omega$$

The system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$



The notion of moment – Linear systems – *Swapped*

The system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

The interpolation point

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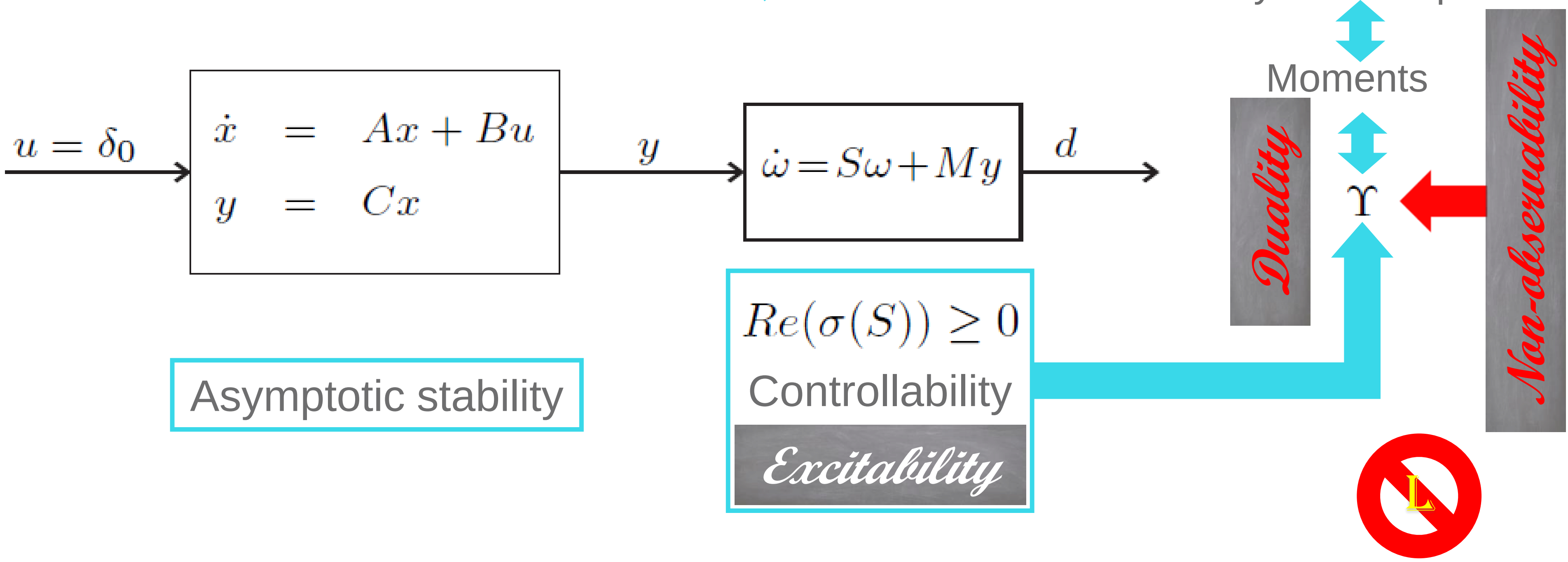
Steady state response

Moments



The notion of moment – Linear systems – Summary

The generator
(interpolation points)

$$\dot{\omega} = S\omega$$

The system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Steady state response

Moments

Υ

Steady state response

Moments

Π

Krylov projectors

$$S\Upsilon = \Upsilon A + MC$$

$$A\Pi + BL = \Pi S$$

A. N. Krylov, On the numerical solution of the equation by which, in technical questions, frequencies of small oscillations of material systems are determined, Izv. Akad. Nauk SSSR, ser. fis.-mat., 1931

The notion of moment – Linear systems – Questions

Is the interconnection approach adequate to extend the notion of moment to nonlinear systems?

How do we interpret the non-observability condition?

Can we provide an intrinsic interpretation of the swapped interconnection without using duality?

Can we simultaneously interconnect generators left and right?

How do we interpret the excitability condition?

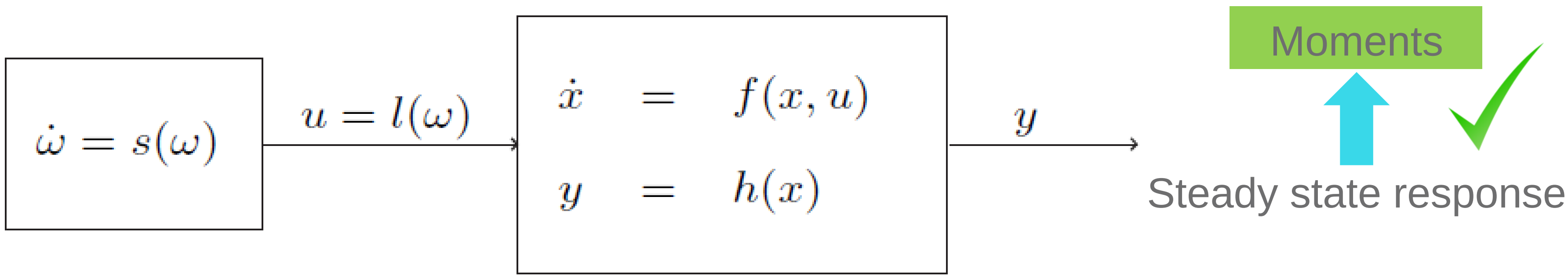
The notion of moment – Nonlinear systems

The *interpolation* point

$$\begin{aligned}\dot{\omega} &= s(\omega)\\ \theta &= l(\omega)\end{aligned}$$

The system

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned}$$



Poisson stability
Observability
Excitability

Asymptotic stability

The notion of moment – Nonlinear systems

The *interpolation* point

$$\begin{aligned} \dot{\omega} &= s(\omega) \\ \theta &= l(\omega) \end{aligned}$$

The system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

$$\dot{\omega} = s(\omega) \quad \xrightarrow{u = l(\omega)} \quad \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \quad \xrightarrow{y}$$

Steady state response

Moments

$$h(\pi(\omega))$$

$$f(\pi(\omega), l(\omega)) = \frac{\partial \pi}{\partial \omega} s(\omega)$$

The notion of moment – Nonlinear systems

The *interpolation* point

$$\begin{aligned} \dot{\omega} &= s(\omega) \\ \theta &= l(\omega) \end{aligned}$$

The system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

The signal generator captures the requirement that one is interested in studying the behaviour of the system only in specific *circumstances*

The interconnected system possesses an invariant manifold and the dynamics restricted to the manifold are a copy of the dynamics of the *signal generator*

$h(\pi(\omega))$ is <u>by definition</u> the moment of the nonlinear system at $s(\omega)$

The notion of moment – Nonlinear systems – *Swapped*

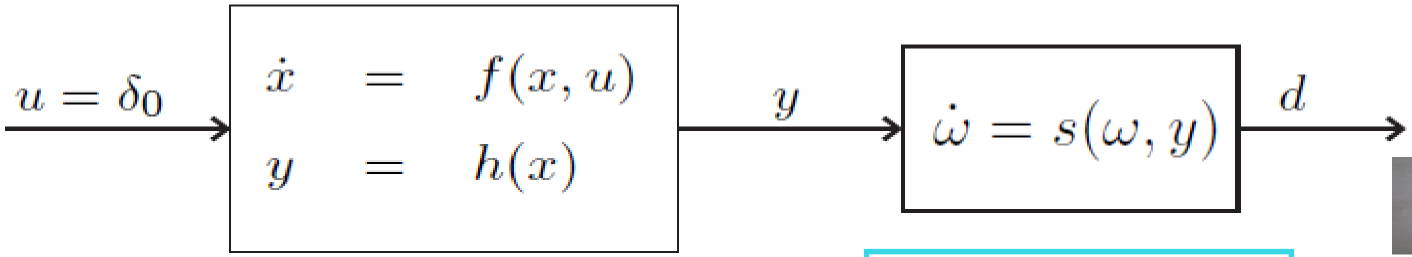
The interpolation point

$$\dot{\omega} = s(\omega, y)$$

The system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned}$$

Moments



Non-observability

Asymptotic stability

CIBS
Poisson stability
Controllability
Excitability

The notion of moment – Nonlinear systems – *Swapped*

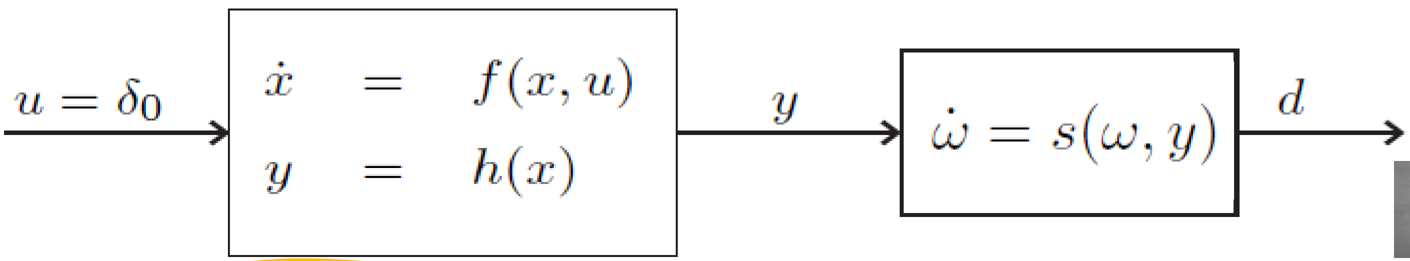
The system

$$\dot{x} = f(x,u)$$
$$y = h(x)$$

The interpolation points

$$\dot{\omega} = s(\omega, y)$$
$$\omega(0) = 0$$

Moments



Non-observability

With special "care"

$$\frac{\partial \rho}{\partial x}\frac{\partial f}{\partial u}$$

$$\rho \circ \pi(\cdot) = \mathrm{Id}$$

$$f(\pi(\omega), l(\omega)) = \frac{\partial \pi}{\partial \omega} s(\omega)$$

The notion of moment – Linear systems – Questions

✓ *Is the interconnection approach adequate to extend the notion of moment to nonlinear systems?*

How do we interpret the non-observability condition?

Can we provide an intrinsic interpretation of the swapped interconnection without using duality?

Can we simultaneously interconnect generators left and right?

How do we interpret the excitability condition?

Before answering these questions we take a detour into circuits theory

Phasor transform and moments

The phasor transform *coincides* with the Sylvester equation defining the moment (at the phasor angular frequency)

The component of the matrix Π are the phasors of the currents and of the integrals of the currents

The phasor of the output response is the moment of the system (at the phasor angular frequency)

A link between the phasor transform and moments:
to be used for *nonlinear* circuits and switched mode electronics

K. A. R. Steinmetz & E. J. Berg, "Complex Quantities and Their Use in Electrical Engineering". Proceedings of the International Electrical Congress, Chicago, American Institute of Electrical Engineers. 1893

Phasor transform and basic circuit elements

$$v = L\frac{di}{dt} \qquad v = \frac{1}{C}\int_0^t i\,d\tau \qquad v = Ri$$

Phasor transform

$$\overline{V} = j\omega L\overline{I} \qquad \overline{V} = \frac{1}{j\omega C}\overline{I} \qquad \overline{V} = R\overline{I}$$

Discontinuous phasor transform

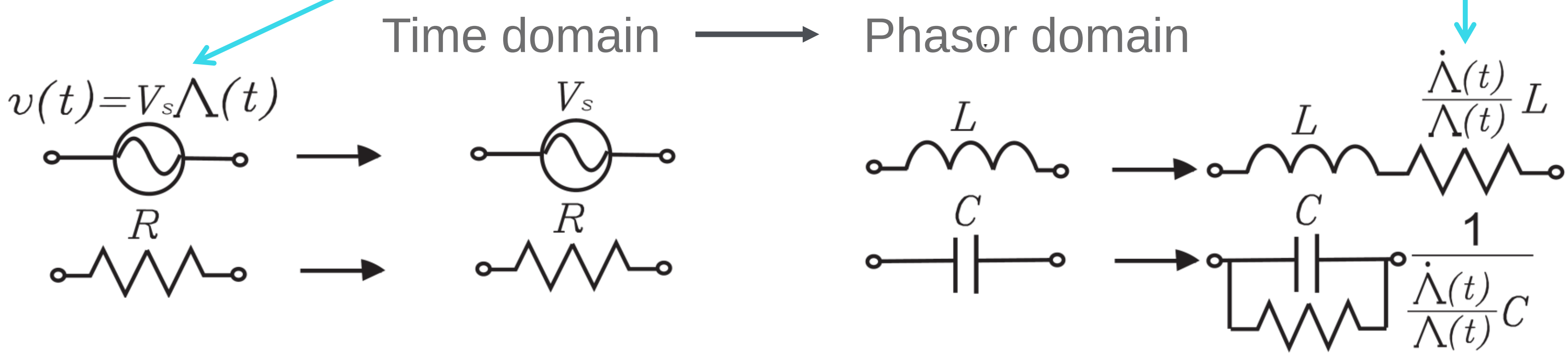
$$\overline{V}(t) = L\dot{\overline{I}}(t) + L\frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{I}(t)$$

$$\dot{\overline{V}}(t) + \frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{V}(t) = \frac{1}{C}\overline{I}(t)$$

It contains information on the “regime”

Phasor transform and circuit elements/properties

It contains information on the “regime”



Average power

Reactive power

$$P = \; < p(t) > = \frac{1}{2}\left\langle \Re\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right]\right\rangle$$

$$Q = \frac{1}{2}\left\langle \Im\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right]\right\rangle$$

Switched mode operation with 50% duty cycle

$$P = \tfrac{1}{4}\Re\left[\overline{V}(t)^*\overline{I}(t)\right]$$

$$Q = \tfrac{1}{4}\Im\left[\overline{V}(t)^*\overline{I}(t)\right]$$

½ in the sinusoidal case

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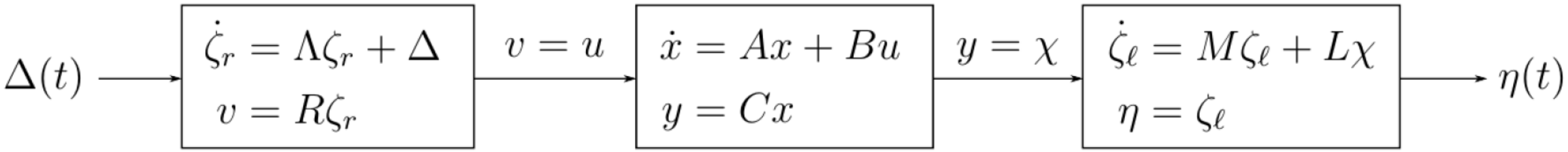
... back to the plan ...

1. *Moments and phasors*
2. *The Loewner functions*
3. *Persistence of excitation*
4. *Adaptive control*
5. *Optimal control*

The notion of moment – Double-sided interconnection



$$\mathbb{L} = \begin{bmatrix} \frac{v_1 r_1 - \ell_1 w_1}{\mu_1 - \lambda_1} & \cdots & \frac{v_1 r_\rho - \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \frac{v_v r_1 - \ell_v w_1}{\mu_v - \lambda_1} & \cdots & \frac{v_v r_\rho - \ell_v w_\rho}{\mu_v - \lambda_\rho} \end{bmatrix}$$

The Loewner (divided difference) matrix

C. Loewner, Über monotone Matrixfunktionen, Mathematische Zeitschrift, 1938.

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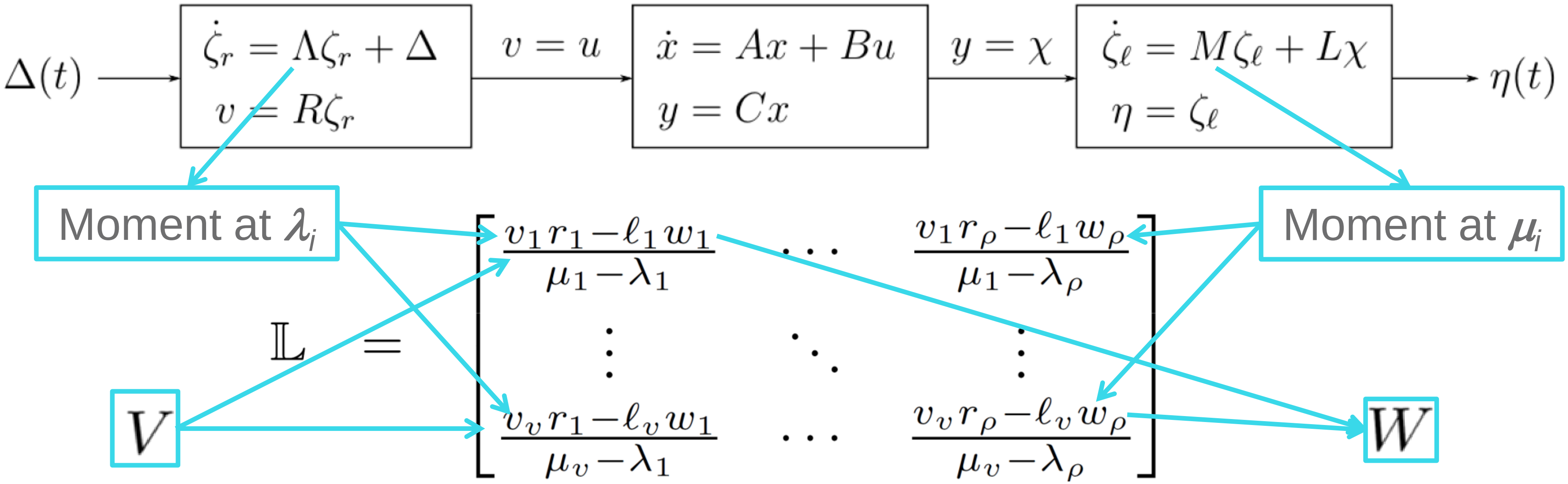
The notion of moment – Double-sided interconnection

The *right (!)* interpolation points

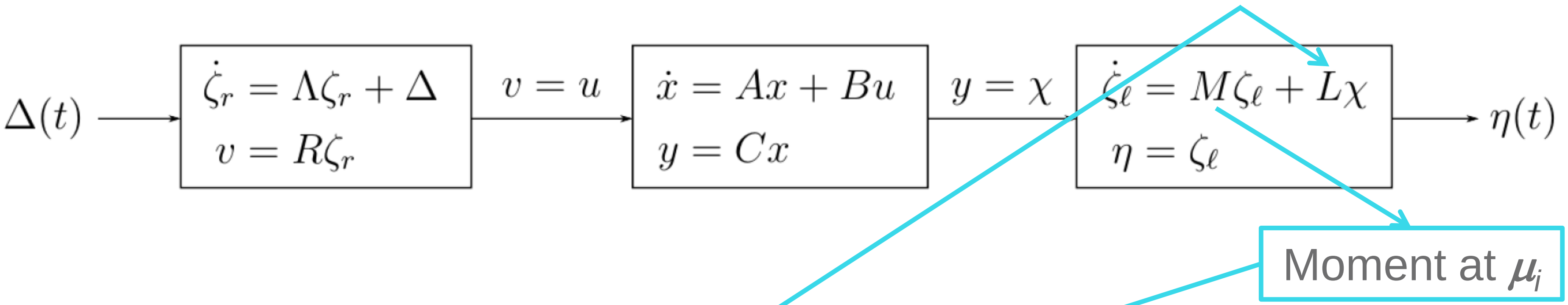
The *left (!)* interpolation points



$$\mathbb{L} = -\begin{bmatrix} \ell_1 C(\mu_1 I - A)^{-1} \\ \vdots \\ \ell_v C(\mu_v I - A)^{-1} \end{bmatrix} \begin{bmatrix} (\lambda_1 I - A)^{-1} B r_1 & \cdots & (\lambda_\rho I - A)^{-1} B r_\rho \end{bmatrix}$$

The notion of moment – Double-sided interconnection

The *right (!)* interpolation points | The *left (!)* interpolation points



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The notion of moment – Double-sided interconnection

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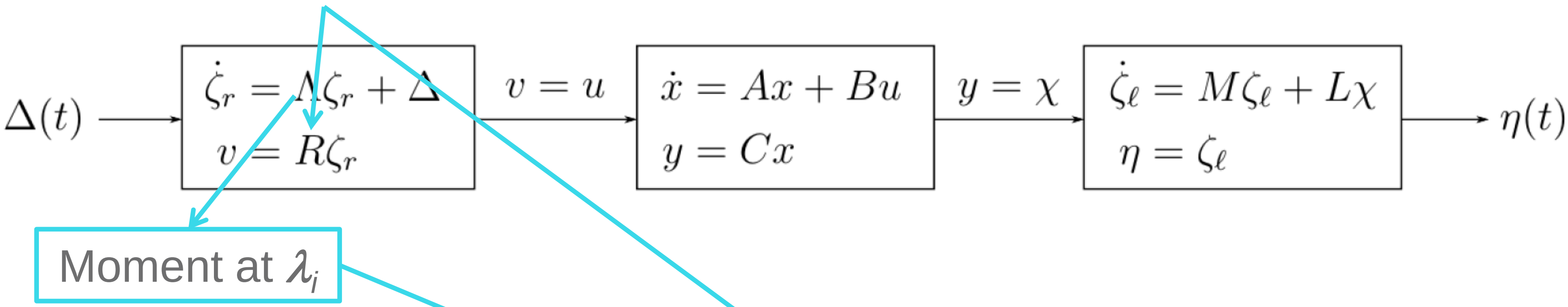
The *left (!)* interpolation points



$$\mathbb{L} = -\begin{bmatrix} \ell_1 C(\mu_1 I - A)^{-1} \\ \vdots \\ \ell_v C(\mu_v I - A)^{-1} \end{bmatrix} \begin{bmatrix} (\lambda_1 I - A)^{-1} B r_1 & \cdots & (\lambda_\rho I - A)^{-1} B r_\rho \end{bmatrix}$$

The notion of moment – Double-sided interconnection

The *right (!)* interpolation points

The *left (!)* interpolation points

$$\Delta(t) \longrightarrow \boxed{\begin{array}{l}\dot{\zeta}_r = \Lambda\zeta_r + \Delta \\ v = R\zeta_r\end{array}} \xrightarrow{v=u} \boxed{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx\end{array}} \xrightarrow{y=\chi} \boxed{\begin{array}{l}\dot{\zeta}_\ell = M\zeta_\ell + L\chi \\ \eta = \zeta_\ell\end{array}} \longrightarrow \eta(t)$$

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR$$

$$\mathbb{L} = -YX$$

Generalized reachability and observability matrices

$$\mathbb{L} = -\begin{bmatrix}\ell_1 C(\mu_1 I - A)^{-1} \\ \vdots \\ \ell_v C(\mu_v I - A)^{-1}\end{bmatrix}\begin{bmatrix}(\lambda_1 I - A)^{-1}Br_1 & \cdots & (\lambda_\rho I - A)^{-1}Br_\rho\end{bmatrix}$$

The notion of moment – Double-sided interconnection

The Loewner matrix allows double-sided interpolation, but heavily relies on the transfer function and on linearity

The associated Sylvester equation is not related to any invariance condition

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR$$

The notion of moment – Double-sided interconnection

The Loewner matrix allows double-sided interpolation, but heavily relies on the transfer function and on linearity

The associated Sylvester equation is not related to any invariance condition

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR$$

$$\mathbb{L} = \mathbb{L}^{\ell} + \mathbb{L}^{r}$$

$$M\mathbb{L}^{\ell} - \mathbb{L}^{\ell}\Lambda \quad = \quad VR \qquad\qquad \mathbb{L}^{r}\Lambda - M\mathbb{L}^{r} \quad = \quad LW$$

✓ Two invariance-like equations

The notion of moment – Double-sided interconnection

$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r$$

$$M\mathbb{L}^\ell - \mathbb{L}^\ell\Lambda = VR \qquad \mathbb{L}^r\Lambda - M\mathbb{L}^r = LW$$

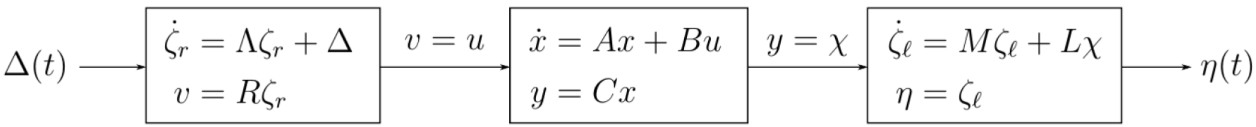
$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR$$

$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r$$

$$\sigma\mathbb{L}^\ell = M\mathbb{L}^\ell \qquad \sigma\mathbb{L}^r = \mathbb{L}^r\Lambda$$

The *shifted left and shifted right Loewner matrices as Lie derivatives* “along the interpolation points”

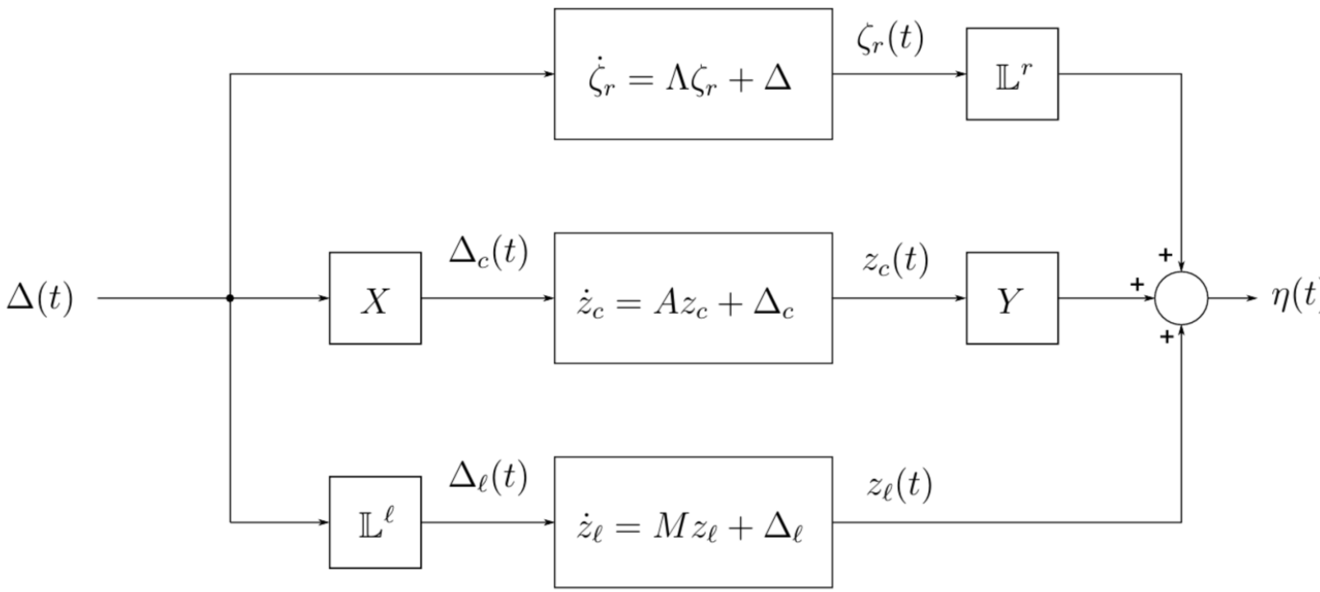
The notion of moment – Double-sided interconnection



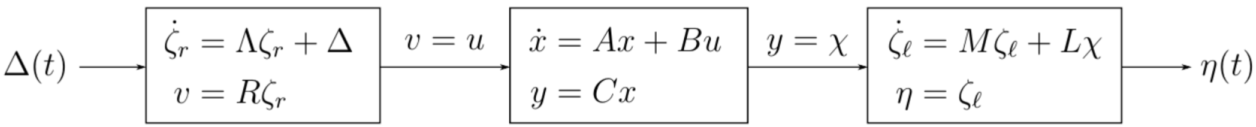
$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r$$

$$M\mathbb{L}^\ell - \mathbb{L}^\ell\Lambda = VR \qquad \mathbb{L}^r\Lambda - M\mathbb{L}^r = LW$$

The left- and right-Loewner matrices *transform* the cascade into a parallel interconnection



The notion of moment – Double-sided interconnection



$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r$$

$$\sigma\mathbb{L}^\ell = M\mathbb{L}^\ell \qquad \sigma\mathbb{L}^r = \mathbb{L}^r\Lambda$$

The Loewner matrices
define the interpolating system

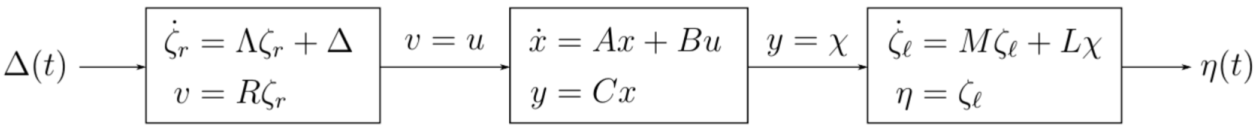
$$\dot{r} = \mathbb{L}^{-1}\sigma\mathbb{L}r - \mathbb{L}^{-1}Vu_r$$

$$y_r = Wr$$

Moment at λ_i

Moment at μ_i

The notion of moment – Double-sided interconnection



$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r$$

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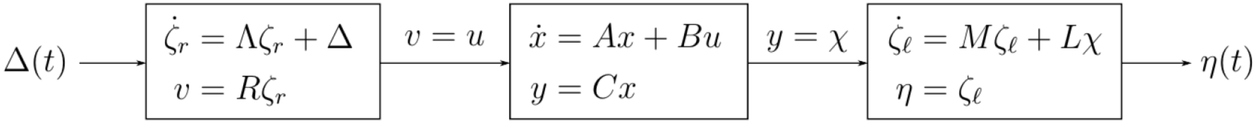
The Loewner matrices
define the interpolating system

$$\dot{r} = \mathbb{L}^{-1}\sigma\mathbb{L}r - \mathbb{L}^{-1}Vu_r$$

$$y_r = Wr$$

The autonomous behaviour of the interpolating system is such that the *shift* and the *time differentiation* commute

The notion of moment – Double-sided interconnection



$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r$$

$$\sigma\mathbb{L}^\ell = M\mathbb{L}^\ell \qquad\qquad \sigma\mathbb{L}^r = \mathbb{L}^r\Lambda$$

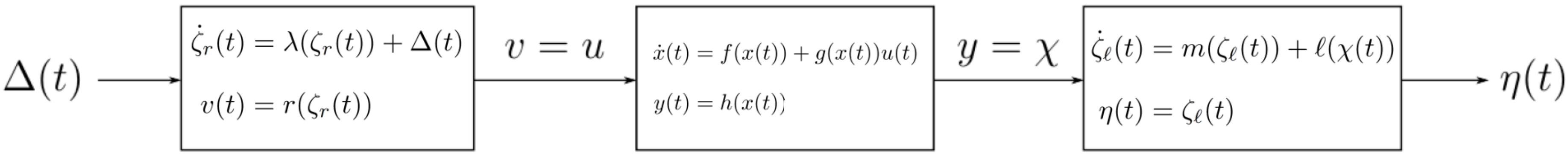
The *shifted left and shifted right Loewner matrices define the interpolating system*

$$\dot{r} = \mathbb{L}(t)^{-1}\left(\sigma\mathbb{L}(t) - \frac{d\mathbb{L}^\ell}{dt}\right)r - \mathbb{L}(t)^{-1}V(t)u_r$$

$$y_r = W(t)r.$$

The autonomous behaviour of the interpolating system is such that the *shift* and the *time differentiation* commute with some *correction*

The notion of moment – Double-sided interconnection



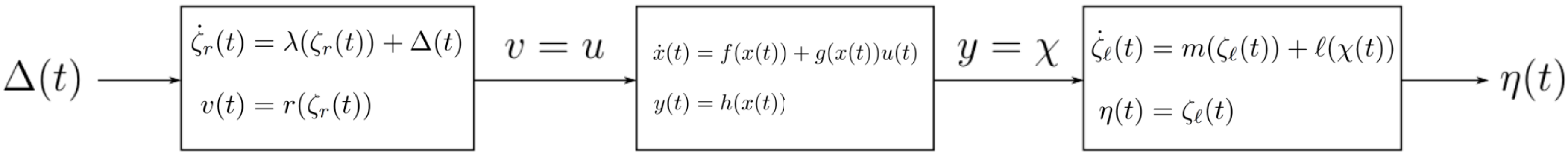
Two (?) invariance-like equations

The left- and right- Loewner "*functions*" *transform* the cascade into a parallel interconnection

The *shifted left and shifted right Loewner "functions"* are *Lie derivatives* "along the interpolation points"

The autonomous behaviour of the interpolating system is such that the *shift* and the *time differentiation* commute with some *correction*

The notion of moment – Double-sided interconnection



$$\frac{\partial X}{\partial \zeta_r}\lambda(\zeta_r) = f(X(\zeta_r)) + g(X(\zeta_r))r(\zeta_r)$$

Three invariance-like equations

$$\frac{\partial Y}{\partial x} f(x) = -m(-Y(x)) - \ell(h(x))$$

$$\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) = -m(-\mathbb{L}^\ell(\zeta_r)) - V(\zeta_r)r(\zeta_r)$$

The left- and right- Loewner "*functions*" *transform* the cascade into a parallel interconnection

$$\mathbb{L}(\zeta_r) := -Y(X(\zeta_r))$$

$$\mathbb{L}^r(\zeta_r) := \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r)$$

$$\sigma\mathbb{L}(\zeta_r) := -m(-\mathbb{L}^\ell(\zeta_r)) + \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r)$$

$$W(\zeta_r) := h(X(\zeta_r))$$

The *shifted left and shifted right Loewner "functions"* are *Lie derivatives* "along the interpolation points"

$$V(\zeta_r) := \frac{\partial Y}{\partial x}(X(\zeta_r))g(X(\zeta_r))$$

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\dot{r} = \sigma\mathbb{L}(r) - V(r)u_r \qquad y_r = W(r)$$

The autonomous behaviour of the interpolating system is such that the *shift* and the *time differentiation* commute with some *correction*

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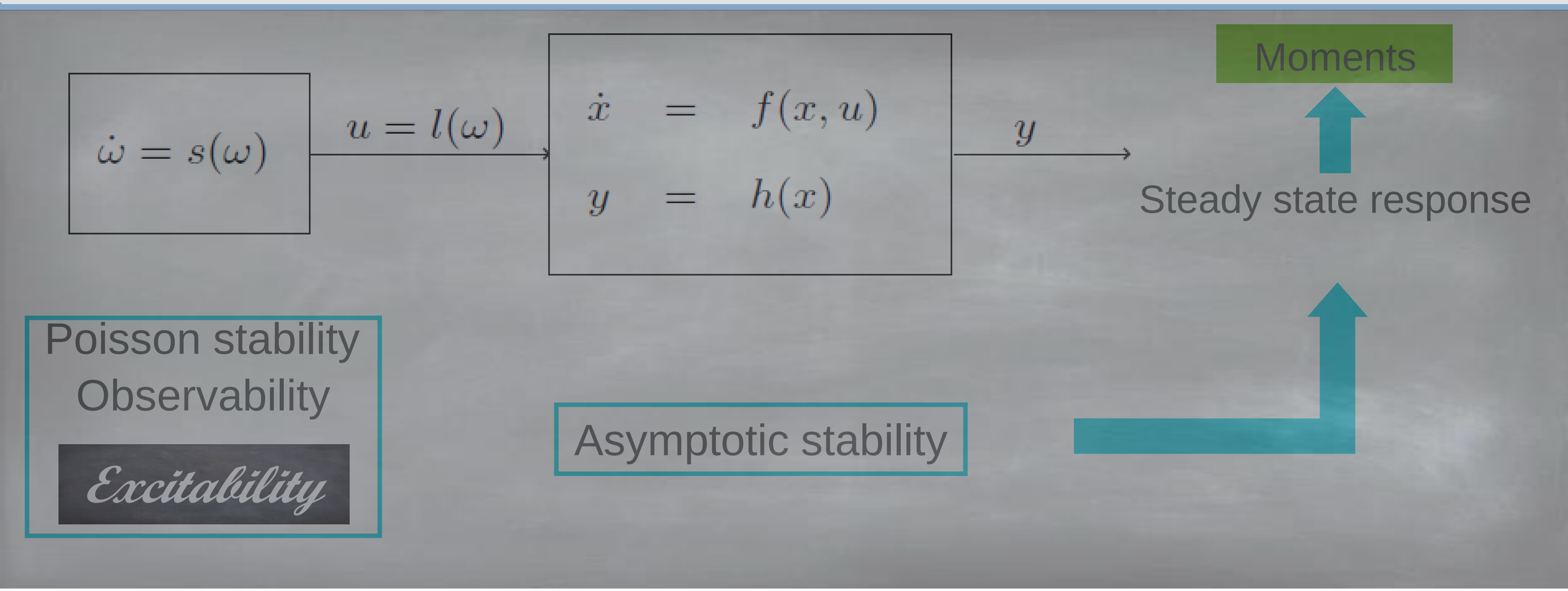
… making progress …

1. Moments and phasors
2. The Loewner functions
3. Persistence of excitation
4. Adaptive control
5. Optimal control

Excitability



$$\dot\omega = S\omega$$
$$\omega(0) = \omega_0$$

Excitability

$$\dot\eta = S\eta + \omega_0 u$$

Reachability

Excitability – Linear systems

$$\dot{\omega} = S\omega$$
$$\omega(0) = \omega_0$$

Excitability

$$\dot{\eta} = S\eta + \omega_0 u$$

Reachability

The excitation *distribution*

$$E(\omega) = \operatorname{span}\left\{S^k \omega,\ k \in \mathbb{Z}_{\geq 0}\right\}$$

The excitation rank condition

$$\dim E(\omega) = \nu \qquad \omega \in W$$

The excitation distribution and the excitation rank condition do not rely on linear concepts (such as the Cayley-Hamilton theorem)

Excitability – Nonlinear systems

$$\dot{\omega} = s(\omega)$$
$$\omega(0) = \omega_0$$

$$\dot{\eta} = s(\eta) + \omega_0 u$$

The excitation *distribution*

$$\theta_{k+1} : W \to W,\ \omega \mapsto \frac{\partial \theta_k}{\partial \omega}(\omega) s(\omega), \quad k \in \mathbb{Z}_{\geq 0}$$

$$E(\omega) = \text{span}\left\{\theta_k(\omega),\ k \in \mathbb{Z}_{\geq 0}\right\}$$

Excitation distribution ≠ Strong accessibility distribution

Excitability – Nonlinear systems

$$\dot{\omega} = s(\omega)$$
$$\omega(0) = \omega_0$$

$$\dot{\eta} = s(\eta) + \omega_0 u$$

The excitation *distribution*

$$\theta_{k+1} : W \to W,\ \omega \mapsto \frac{\partial \theta_k}{\partial \omega}(\omega) s(\omega), \quad k \in \mathbb{Z}_{\geq 0}$$

$$E(\omega) = \operatorname{span}\{\theta_k(\omega),\ k \in \mathbb{Z}_{\geq 0}\}$$

The excitation rank condition

$$\dim E(\omega) = \nu \qquad \omega \in W$$

Excitation rank condition ≠ Strong accessibility rank condition

Excitability – A geometric characterization

$$\dot{\omega} = s(\omega)$$
$$\omega(0) = \omega_0$$

Suppose solutions are analytic

Solution is *PE* ⟺

$$\mathcal{W}_{[t,t+T]} = \int_t^{t+T} \omega(\tau)\omega(\tau)^\top d\tau$$

PD for all t *and some* T

⟺ The excitation rank condition holds at every $\omega \in \gamma_+(\omega_0)$

✓

Excitability – A geometric characterization

$$\dot{\omega} = s(\omega)$$
$$\omega(0) = \omega_0$$

Suppose solutions are analytic and ω_0 is almost periodic

Solution is *PE* ⟺

$$\mathcal{W}_{[t,t+T]} = \int_t^{t+T} \omega(\tau)\omega(\tau)^\top d\tau$$

PD for all t *and some* T

⟺ The excitation rank condition holds at ω_0

H. Bohr, "Zur Theorie der fastperiodischen Funktionen I" Acta Math., 1925

✓

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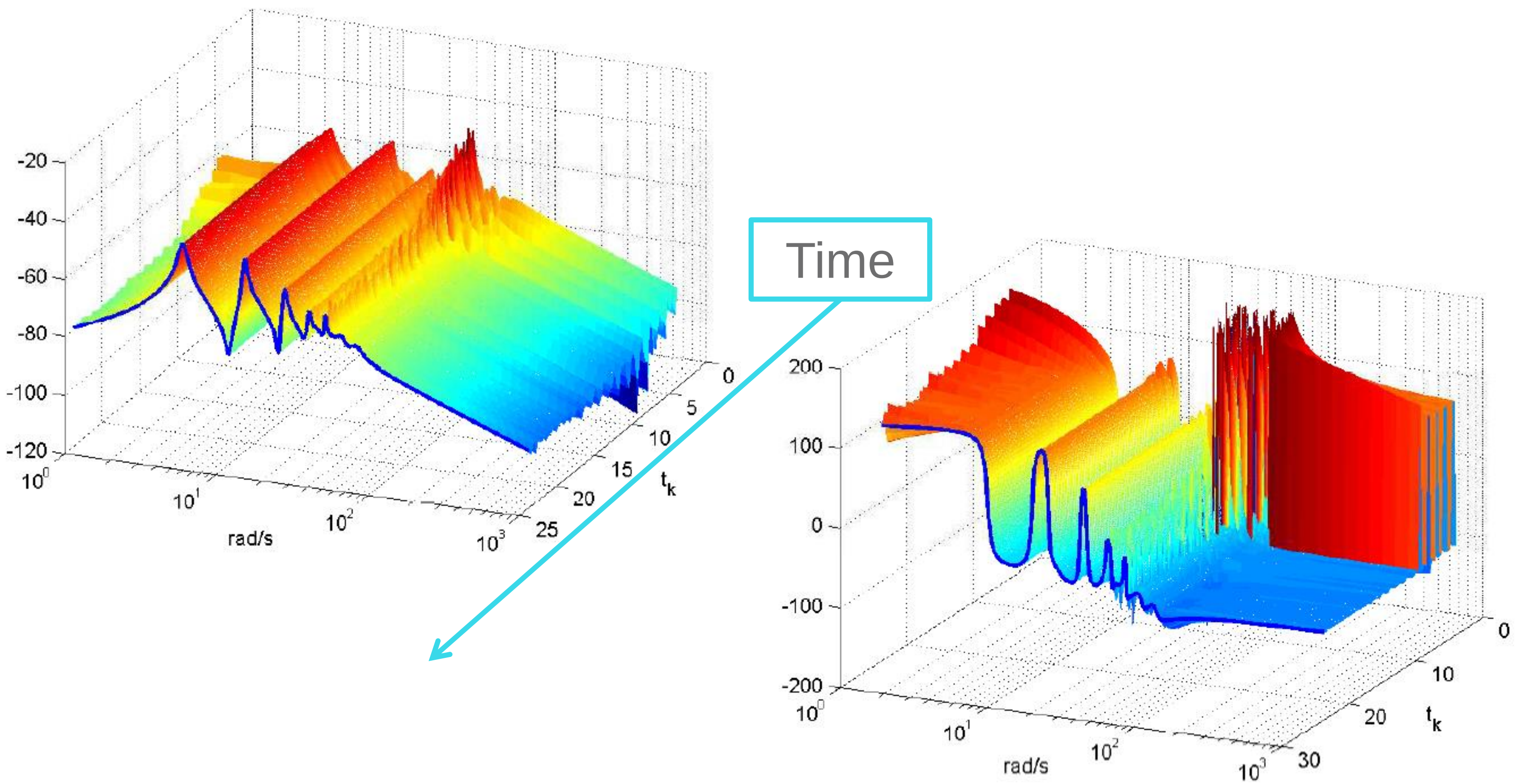
Excitability – Applications

The excitability rank condition plays a fundamental role in model reduction problems from data



Excitability – Applications

The excitability rank condition, hence the PE condition, plays a fundamental role stability analysis of *skew-symmetric* systems

$$\dot{x} = -\varphi\varphi^\top x \qquad \dot{x} = \begin{bmatrix} A & B\varphi^\top \\ -\varphi C & 0 \end{bmatrix} x$$

$$\begin{array}{l} \dot{\omega} = s(\omega) \\ \omega(0) = \omega_0 \end{array} \quad \xrightarrow{\varphi = \omega} \quad \dot{x} = \begin{bmatrix} A & B\varphi^\top \\ -\varphi C & 0 \end{bmatrix} x$$

Almost-periodicity, excitability rank condition, minimality imply uniformly globally exponential stability

A. Cayley, "Sur les determinants gauches", Crelle's Journal, 1847.

... more progress ...

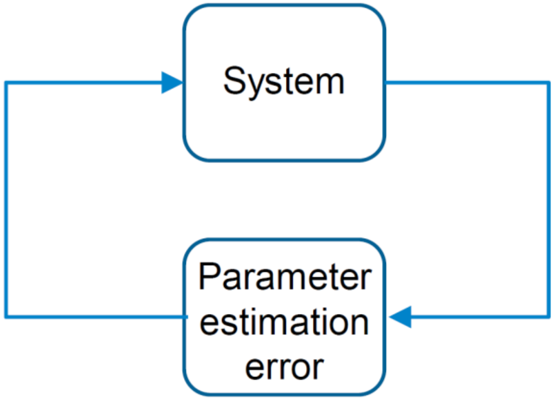
1. ✓ *Moments and phasors*
2. ✓ *The Loewner functions*
3. ✓ *Persistence of excitation*
4. *Adaptive control*
5. *Optimal control*

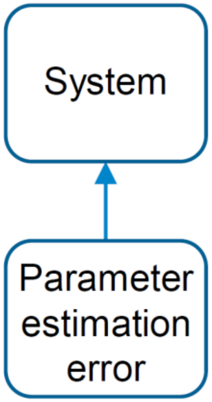
Adaptive control

Excitability and PE naturally lead to adaptive control

Where is the curse of linearity/time-invariance in adaptive control?

Classical adaptive control	I&I adaptive control



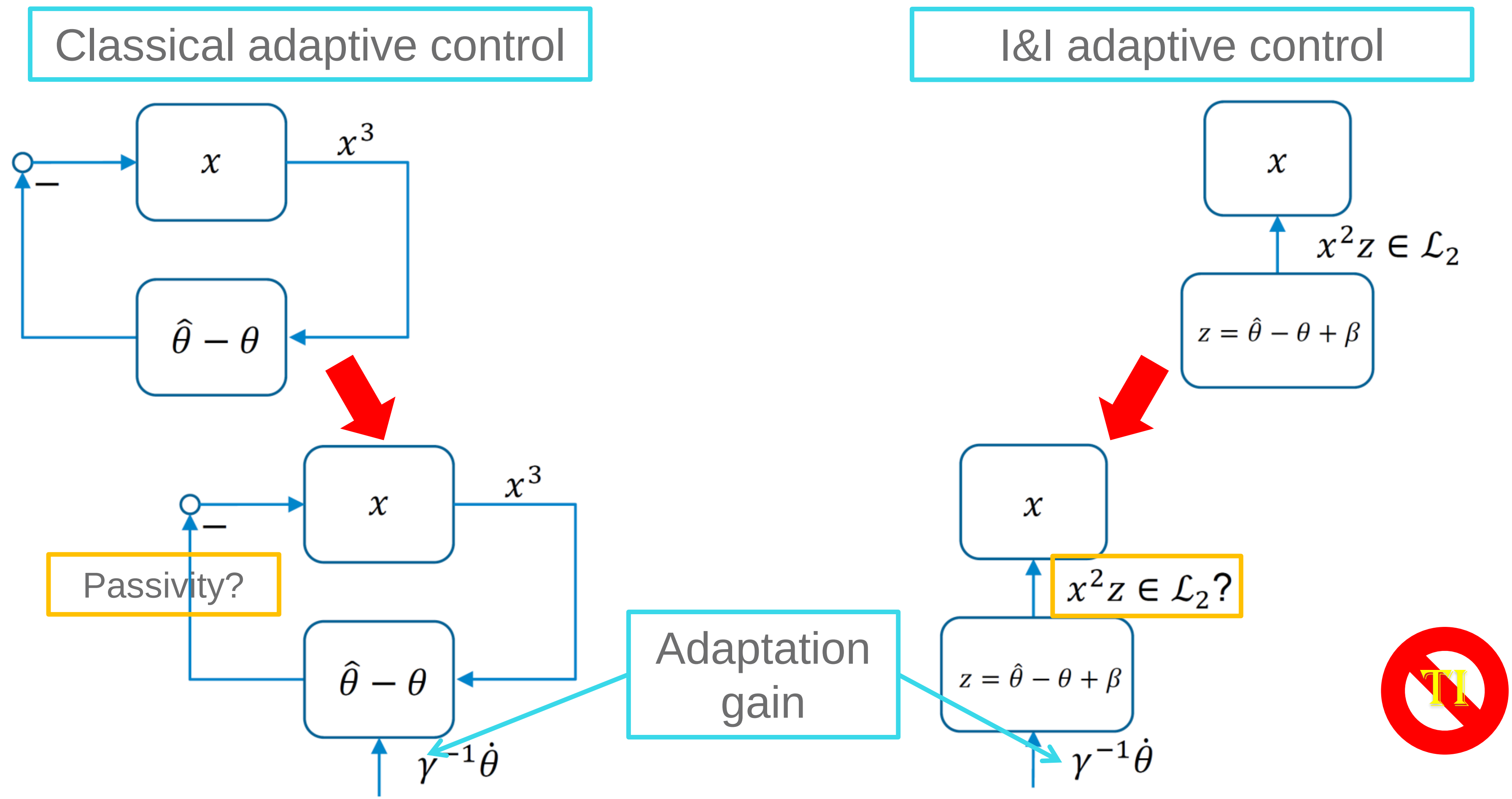


Design the update law to *create* a passive interconnection

Design the update law to *create* an L_2 stable cascade

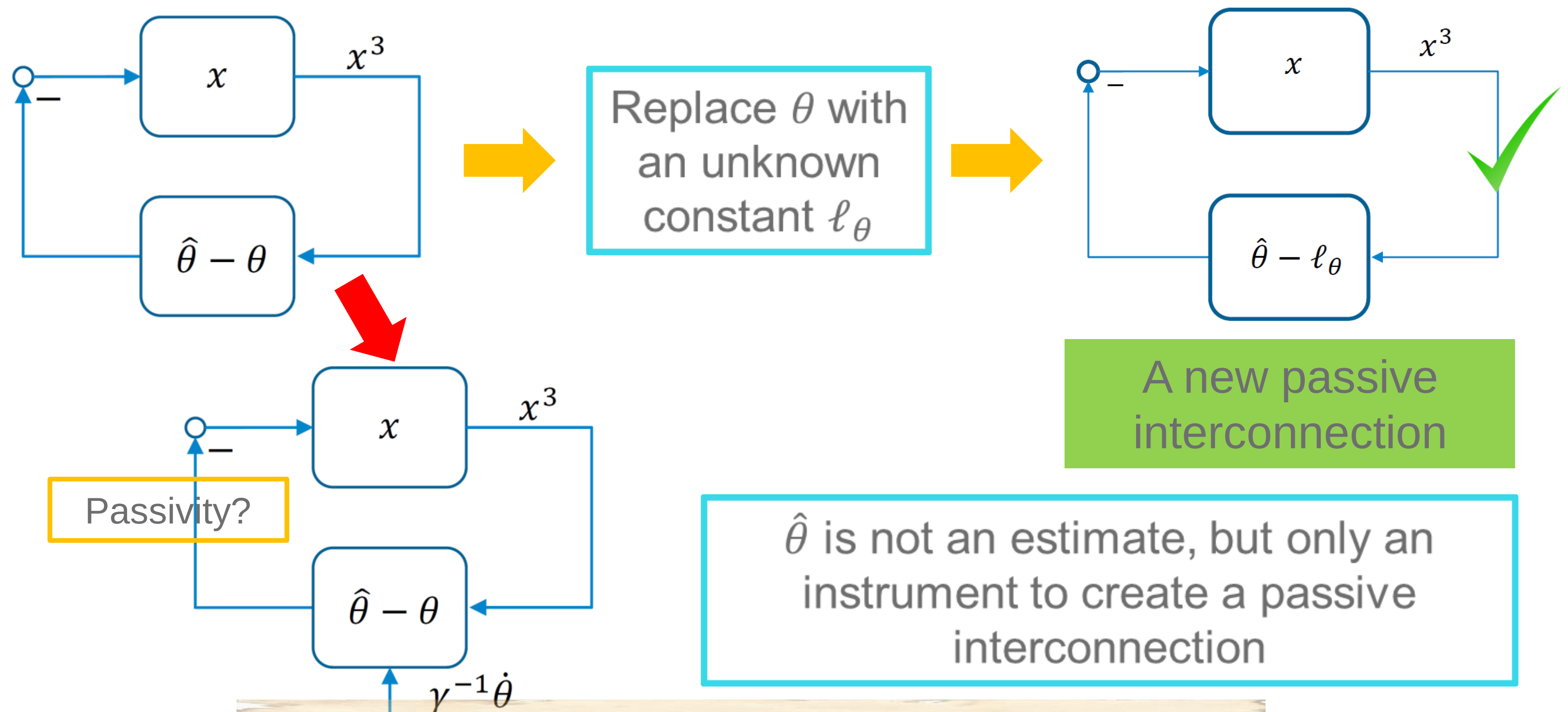
Adaptive control

$$\dot{x} = \theta(t)x^2 + u$$



Adaptive control – Congelation of variables

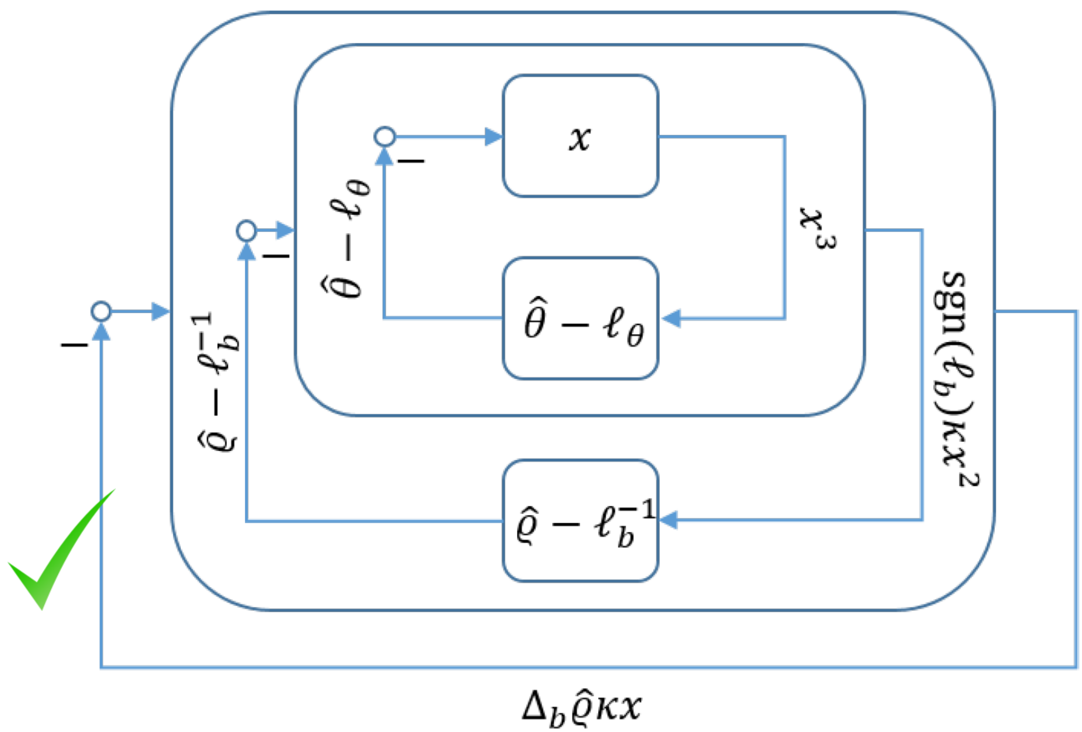
$$\dot{x} = \theta(t)x^2 + u$$



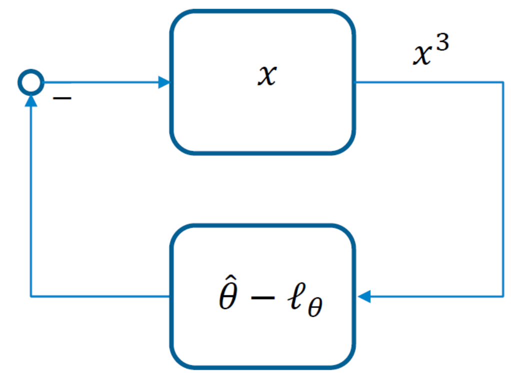
S. Bochner, Vorlesungen über Fouriersche Integrale, 1932

Adaptive control – Congelation of variables

$$\dot{x} = \theta(t)x^2 + b(t)u$$



A new passive interconnection



A new passive interconnection

Adaptive control – Congelation of variables

The congelation of variables allows to recover, in simple cases, passive interconnections. ✓

How do we extend this idea to more general systems?

Is time-invariance exploited in other steps of classical adaptive design?

$$\dot{x}_1 = x_2 + \phi_{0,1}(y) + \sum_{j=1}^{q} \phi_{1,j}(y) a_j(t)$$

$$\dot{x}_\rho = x_{\rho+1} + \phi_{0,\rho}(y) + \sum_{j=1}^{q} \phi_{\rho,j}(y) a_j(t) + b_m(t) g(y) u$$

$$\dot{x}_n = \phi_{0,n}(y) + \sum_{j=1}^{q} \phi_{n,j}(y) a_j(t) + b_0(t) g(y) u$$

$$y = x_1$$

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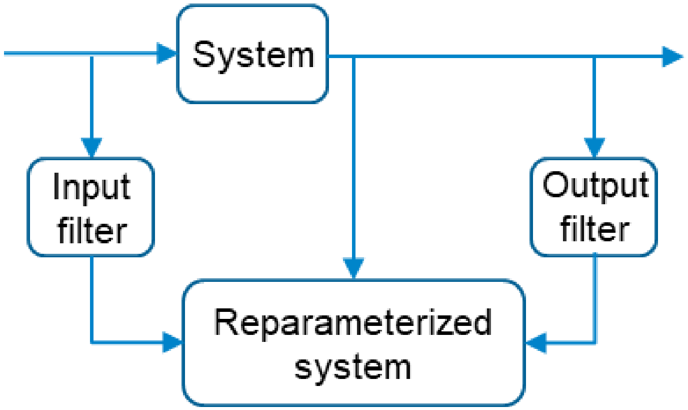
Adaptive control – Congelation of variables

$$\dot{x}_1 = x_2 + \phi_{0,1}(y) + \sum_{j=1}^{q} \phi_{1,j}(y) a_j(t)$$

$$\dot{x}_\rho = x_{\rho+1} + \phi_{0,\rho}(y) + \sum_{j=1}^{q} \phi_{\rho,j}(y) a_j(t) + b_m(t) g(y) u$$

$$\dot{x}_n = \phi_{0,n}(y) + \sum_{j=1}^{q} \phi_{n,j}(y) a_j(t) + b_0(t) g(y) u$$

$$y = x_1$$



Kreisselmeier filters (input, output, regressor)

$$\dot{\varepsilon} = A_k \varepsilon + \Phi^\top(y) \Delta_a + \begin{bmatrix} 0_{(\rho-1)\times 1} \\ \Delta_b \end{bmatrix} g(y) u$$

A congealed change of coordinates

$$x = \xi + [v_m, \dots, v_0, \Xi^\top] \ell_\theta + \varepsilon$$

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Adaptive control – Congelation of variables

The re-parameterized system

$$\begin{aligned}\dot{y} &= \omega_0 + \overline{\omega}\ell_\theta + \varepsilon_2 + \ell_{b_m} v_{m,2}\\ &\vdots\\ \dot{v}_{m,i} &= -k_i v_{m,1} + v_{m,i+1}\\ &\vdots\\ v_{m,\rho} &= -k_\rho v_{m,1} + v_{m,\rho+1} + g(y)u\end{aligned}$$

The inverse dynamics

$$\begin{aligned}\dot{x}_{\rho+1} &= -\frac{b_{m-1}}{b_m}x_{\rho+1} + x_{\rho+2} + y_{\rho+1} + \frac{b_{m-1}}{b_m}\left(y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho\right)\\ &\vdots\\ \dot{x}_n &= -\frac{b_{m-1}}{b_m}x_{\rho+1} + y_n + \frac{b_0}{b_m}\left(y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho\right)\end{aligned}$$

Adaptive control – Congelation of variables

The inverse dynamics

$$\dot{x}_{\rho+1} = -\frac{b_{m-1}}{b_m} x_{\rho+1} + x_{\rho+2} + y_{\rho+1} + \frac{b_{m-1}}{b_m}\left(y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho\right)$$

$$\vdots$$

$$\dot{x}_n = -\frac{b_{m-1}}{b_m} x_{\rho+1} + y_n + \frac{b_0}{b_m}\left(y^{(\rho)} - y_1^{(\rho-1)} - \cdots - y_\rho\right)$$

A change of coordinates, the chain rule and standard backstepping …

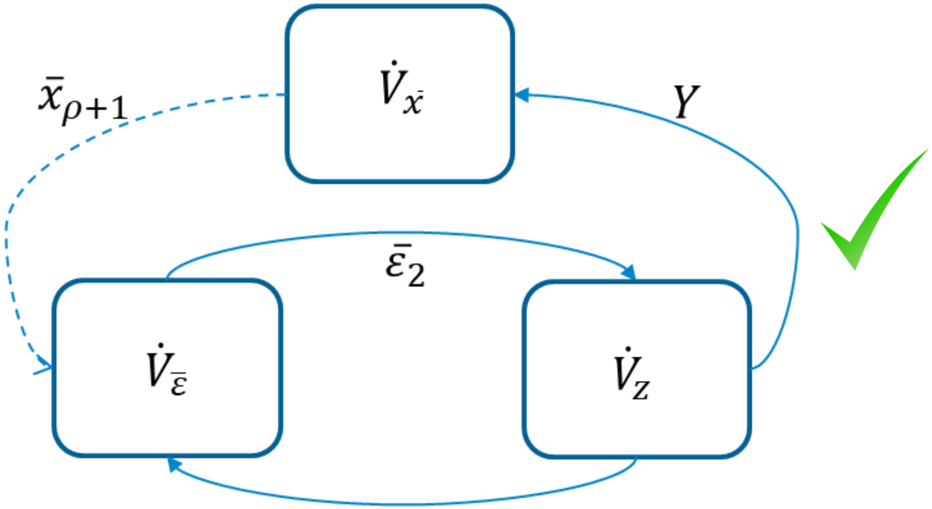
$$\bar{x}_n = x_n - \sum_{j=0}^{\rho-1} (-1)^j \left(\frac{b_0}{b_m}\right)^{(\rho-1-j)} + \sum_{i=1}^{\rho-1} \sum_{j=0}^{\rho-i-1} (-1)^j \left(\frac{b_0}{b_m}\right)^{(\rho-i-1-j)}$$

$$s_1 s_2^{(i)} = (-1)^i s_1^{(i)} s_2 + \left(\sum_{j=0}^{i-1} (-1)^j s_1^{(j)} s_2^{(i-1-j)}\right)^{(1)}$$

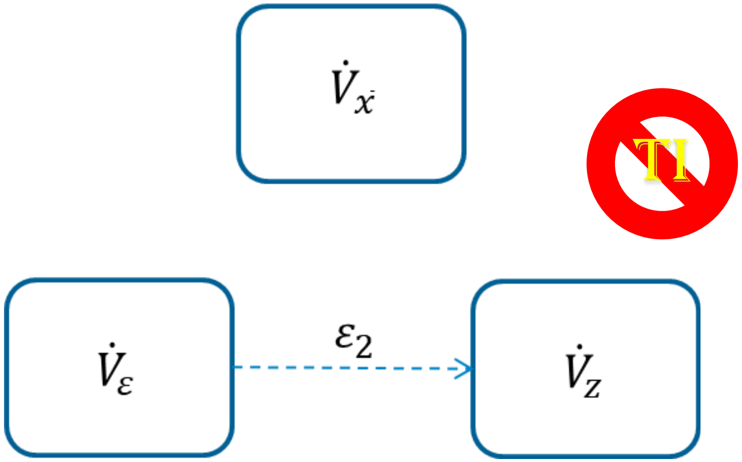
Adaptive control – Congelation of variables

The congelation of variables allows constructing a new interconnection

The time-invariant case does not reveal all interconnections



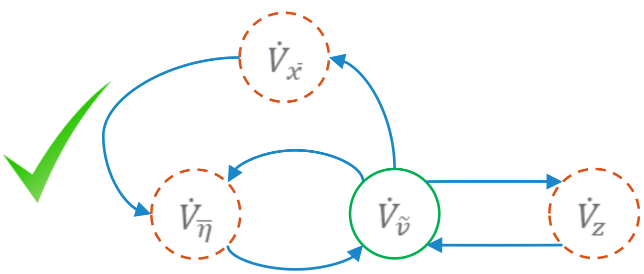
$Y(y)$ *collects* $y,\ \phi_{0,i}(y), \phi_{i,j}(y)$

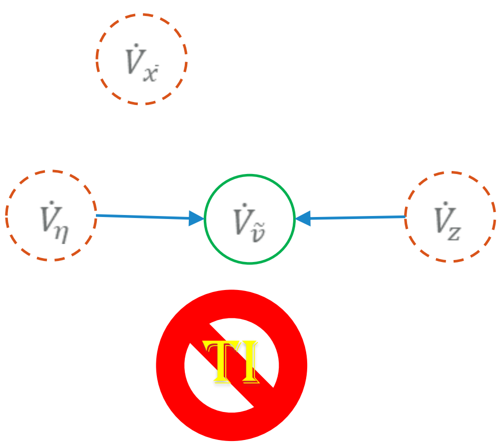


The interconnections are however present whenever parameters vary

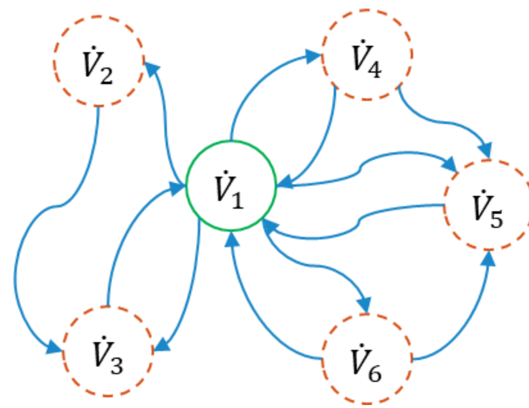
Adaptive control – Congelation of variables

I&I adaptive control





A general stabilization result



$\widehat{\sum_i V_i} \leq 0$ can be achieved by damping and scaling if there is at least one green node in every loop

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... and to conclude

1. *Moments and phasors*
2. *The Loewner functions*
3. *Persistence of excitation*
4. *Adaptive control*
5. *Optimal control*

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Optimal control

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Optimal control problems can be solved using dynamic programming or Pontryagin's minimum principle

Combining both approaches may yield a new perspective and new optimality conditions

The basic *linear* ingredients

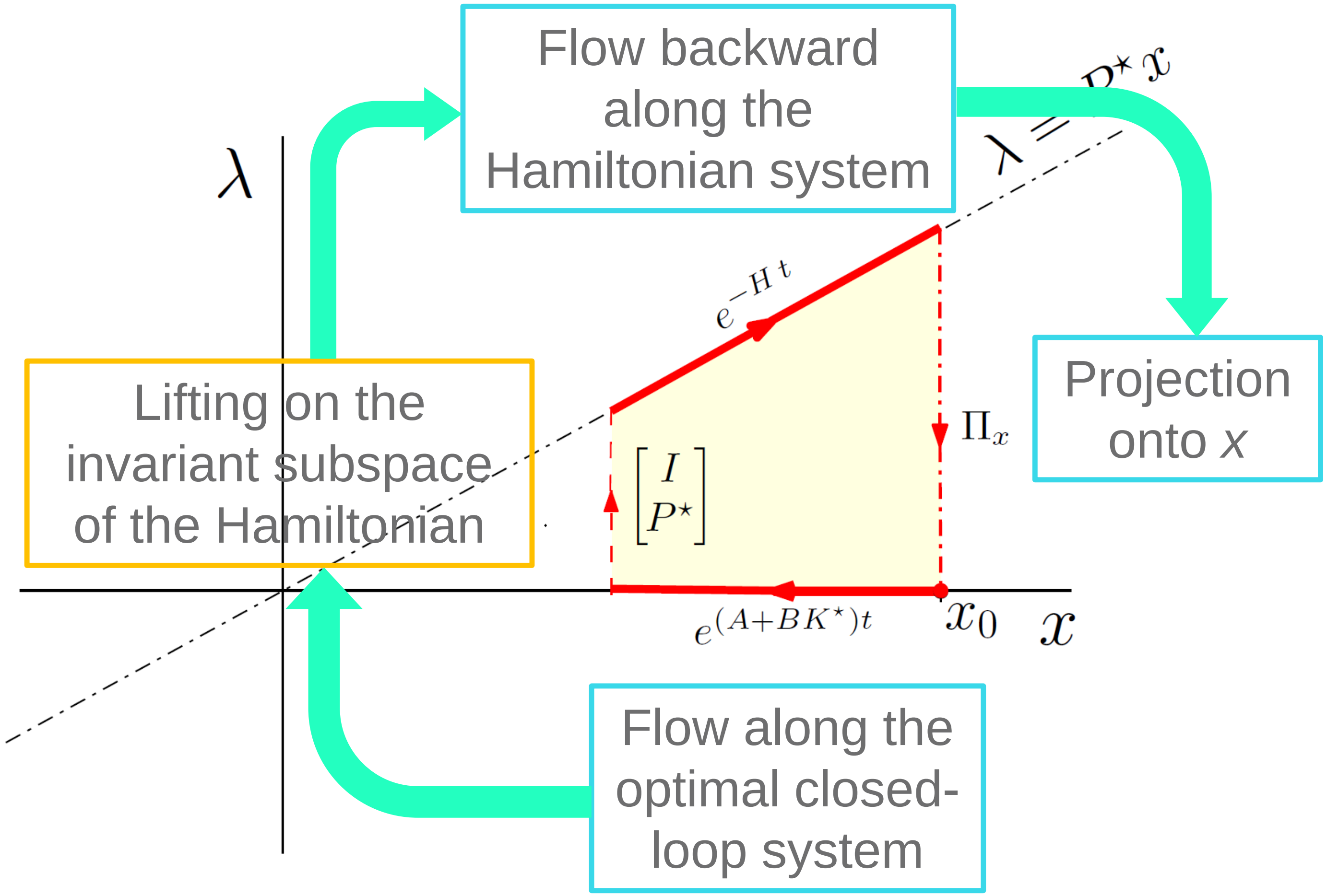
$$\dot{x} = Ax + Bu \qquad J_{x_0}(u) = \frac{1}{2}\int_0^\infty (x(t)^\top Q x(t) + u(t)^\top R u(t))dt$$

$$0 = Q + A^\top P + PA - PBR^{-1}B^\top P \qquad H = \begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}$$

Timaeus of Taormina, „Queen Dido and the isoperimetric problem", c. 345 BC – c. 250 BC

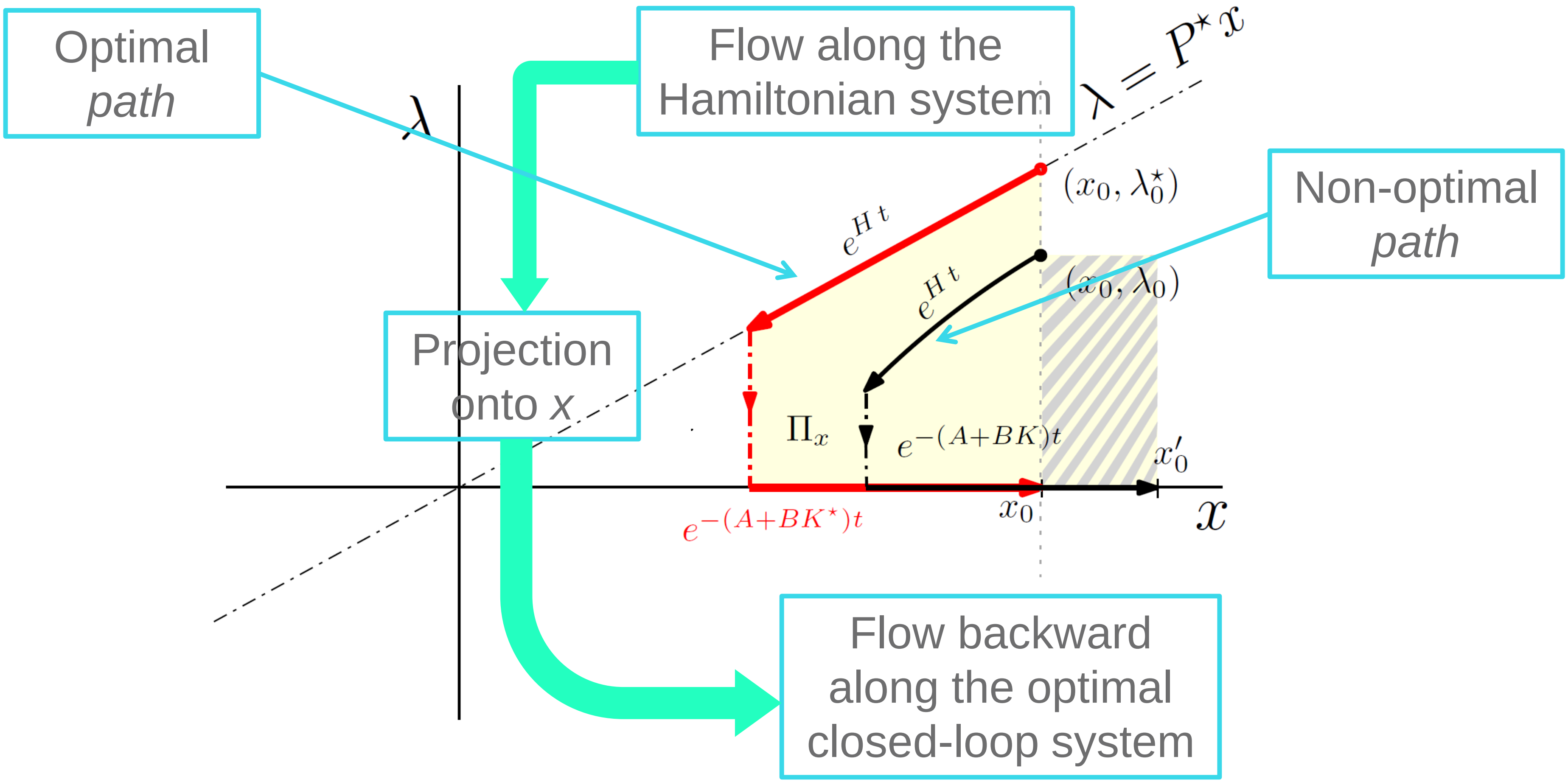
Optimal control – A graphical interpretation

For any initial condition and any *t>0* the composition has a fixed point



Optimal control – A graphical interpretation

… and equivalently



Optimal control – A graphical interpretation

The optimal feedback gain and the solution of the Riccati equation are such that

$$\sigma(A + BK^{\star}) \subset \mathbb{C}^{-}$$

Stability

$$\Upsilon_{P^{\star},K^{\star}}(t, x_0) = x_0$$

Fixed point

$$\Upsilon_{P,K}(t, x_0) \triangleq \Pi_x \left(e^{-Ht} \begin{bmatrix} I_n \\ P \end{bmatrix} e^{(A+BK)t} x_0 \right)$$

⇕

$$\Pi_x \circ \Phi_H(-t; \cdot) \circ \varphi_{(A+BK)x}(t; x_0)$$

Optimal control – Linear systems

The optimal feedback gain and the solution of the Riccati equation are such that

$$\sigma(A + BK^\star) \subset \mathbb{C}^-$$

$$\begin{cases} 0 = MHN(P^\star) - (A + BK^\star) \\ 0 = MH^2N(P^\star) - (A + BK^\star)^2 \\ \vdots \\ 0 = MH^{2n}N(P^\star) - (A + BK^\star)^{2n} \end{cases}$$

Linear in P^*

Polynomial in K^*

This relies on Cayley-Hamilton theorem

Optimal control – Linear systems

The optimal feedback gain and the solution of the Riccati equation are such that

$$\sigma(A + BK^\star) \subset \mathbb{C}^-$$

$$0 = W(K^\star) + (Y - J_2(K^\star))^\top P^\star + P^\star (Y - J_2(K^\star)) + P^\star X P^\star$$

Positive semi-definite

Positive semi-definite, also for robust control problems

This is amenable to the use of optimization algorithms on the manifold of positive definite matrices with cost

$$J = \mathrm{tr}(W + (Y - J_2)^\top P + P(Y - J_2) + PXP)^2$$

Optimal control – Nonlinear systems

The basic *nonlinear* ingredients

$$\dot{x} = f(x) + g(x)u \qquad \min_u \left\{ \frac{1}{2} \int_0^\infty (q(x(t)) + \|u(t)\|^2) dt \right\}$$

$$0 = \frac{1}{2} q(x) + \frac{\partial V}{\partial x}(x) f(x) - \frac{1}{2} \frac{\partial V}{\partial x}(x) g(x) g(x)^\top \frac{\partial V}{\partial x}(x)^\top$$

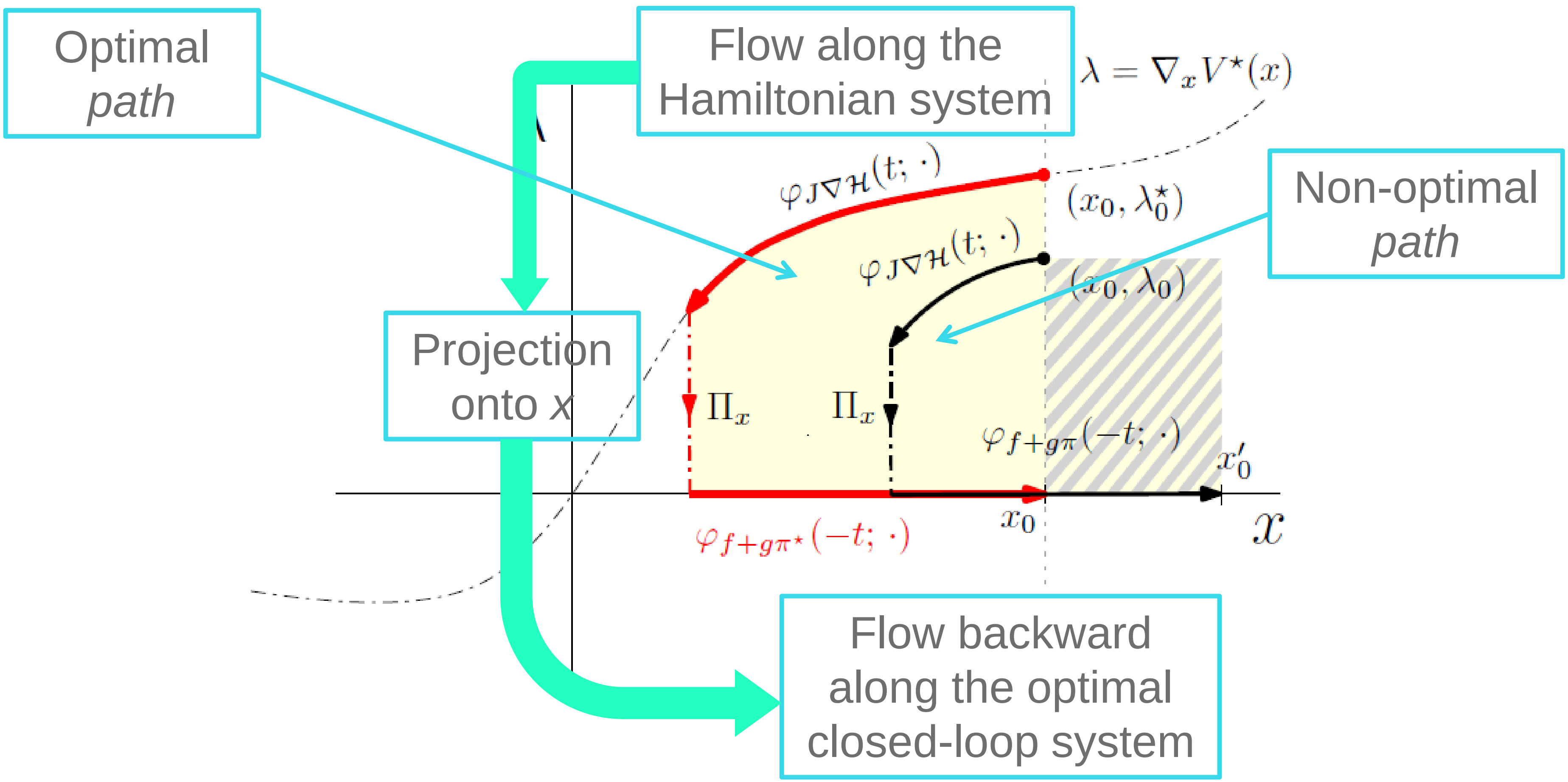
$$\mathcal{H}(x, \lambda) = \frac{1}{2} q(x) + \lambda^\top f(x) - \frac{1}{2} \lambda^\top g(x) g(x)^\top \lambda$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = J \nabla \mathcal{H}(x, \lambda)$$

Skew-symmetric

Optimal control – A graphical interpretation

Exactly the same interpretation as for linear systems



Optimal control – A graphical interpretation

The optimal feedback gain and the solution of the optimal costate equation are such that

Stability

Fixed point

$$\varphi_{f+g\pi}(-t;\Pi_x \circ \varphi_{J\nabla\mathcal{H}}(t;x_0,\lambda_0)) - x_0 = 0$$

$\Updownarrow$

$$0 = M\mathcal{D}_i(J\nabla\mathcal{H})(x_0,\lambda_0) - \mathcal{D}_i(f+g\pi)(x_0)$$

$$\mathcal{D}_i(f) = (\nabla\mathcal{D}_{i-1}(f))\mathcal{D}_0(f)$$

✓ This is amenable to compute arbitrarily accurate approximations of the optimal control law without solving any PDE

Summary

1. *Moments and phasors*
2. *The Loewner functions*
3. *Persistence of excitation*

Analysis

4. *Adaptive control*
5. *Optimal control*

Design

Some take-away messages

1. *Linearity and time-invariance provide a very powerful structure which may be misleading*

2. *A careful study of linear, time-invariant systems with abstract tools allows*
   - *developing nonlinear/time-varying enhancement of analysis and design tools*
   - *improving our understanding of essential features and interactions*
   - *….*
   - *breaking the curse of linearity and time-invariance*

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Thanks!

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The curse of linearity and time-invariance

Alessandro Astolfi
Imperial College London
and
University of Rome "Tor Vergata"

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